ACADEMIC PRESS

Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of Sound and Vibration 267 (2003) 1047–1064

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Interaction of eigenvalues in multi-parameter problems

A.P. Seyranian*, A.A. Mailybaev

Institute of Mechanics, Moscow State Lomonosov University, Michurinsky pr. 1, 119192 Moscow, Russia

Received 8 July 2002; accepted 29 October 2002

Abstract

The paper presents a new general theory of interaction of eigenvalues of matrix operators depending on parameters. Both complex and real eigenvalues are considered. Strong and weak interactions are distinguished, and their geometric interpretation on the complex plane is given. Mechanical examples are presented and discussed in detail.

1. Introduction

Behaviour of eigenvalues with a change of parameters is a problem of general interest for applied mathematicians and natural scientists. This problem has many important applications in aerospace, mechanical, civil, and electrical engineering. Behaviour of eigenvalues depending on parameters is of a special value for vibration and stability problems, see, for example, the books by Bolotin [1], Panovko and Gubanova [2], Ziegler [3], Huseyin [4], Leipholz [5], Thompson [6], Thomsen [7], and Paidoussis [8].

The theory of interaction of eigenvalues, presented below, is based on the constructive perturbation methods, developed mostly by Vishik, Lyusternik, and Lidskii, see Refs. [9–11], and their extension to the multi-parameter case done by Seyranian [12,13], Mailybaev and Seyranian [14], and Seyranian and Kirillov [15]. In Refs. [12,13] the notion of strong and weak interactions of eigenvalues was introduced based on a Jordan form of the system matrix. If there are two eigenvectors corresponding to a double eigenvalue the interaction is called *weak*, and if there exists only one eigenvector with a double eigenvalue the interaction is called *strong*. It was shown that the strong interaction in one-parameter systems is characterized by two parabolae of equal curvature at the intersection point lying in perpendicular planes, while weak interaction occurs in

*Corresponding author. Tel.: +7095-939-5478; fax: +7095-939-0165.
E-mail address: seyran@imec.mus.ru (A.P. Seyranian).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00360-2

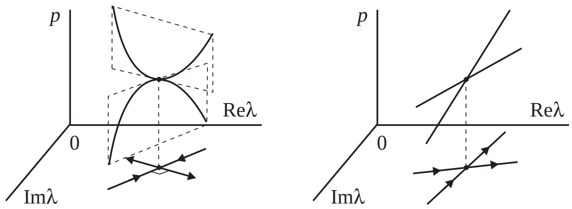

Fig. 1. Strong and weak interactions of eigenvalues in one-parameter problem.

the same plane; see Fig. 1. It is remarkable that the speed $d\lambda/dp$ of strong interaction tends to infinity at the intersection point, while weak interaction is characterized by finite interaction speed.

Some results on interaction of complex eigenvalues in vibrational systems were obtained by Seyranian [16], and Seyranian and Pedersen [17]. In these two papers the important mechanical effects like destabilization of a non-conservative system by infinitely small damping and transference of instability between eigenvalue branches were described and explained. Overlapping of frequency curves in circulatory systems was studied in Kirillov and Seyranian [18], and stability boundaries of Hamiltonian and gyroscopic systems were examined by Mailybaev and Seyranian [19] and Seyranian and Kliem [20].

In this paper, a general theory of interaction of two eigenvalues of matrix operators on the complex plane depending on multiple parameters is presented. Both complex and real eigenvalues are considered. Strong and weak interactions of eigenvalues, when one of the parameters is changed while increments of others remain constant, are distinguished based on a Jordan form of the system matrix. It is shown that the strong interaction of eigenvalues on the complex plane is described by hyperbolae with perpendicular asymptotes, if the double eigenvalue λ_0 is complex, and by a parabola and straight line, if λ_0 is real. Weak interaction of eigenvalues, characterized by two linearly independent eigenvectors at the point of coincidence, is studied. It is revealed that weak interaction is described by hyperbolae or a small elliptic bubble appearing from the point of the double eigenvalue λ_0 perpendicular to the plane of original interaction in the case of a real λ_0. And if λ_0 is complex the interacting eigenvalues keep or interchange their main directions of motion on the complex plane before and after the weak interaction. It is emphasized that the presented theory of interactions gives not only qualitative, but also quantitative results on behaviour of eigenvalues based only on the information at the initial point in the parameter space.

As mechanical examples, vibrational systems with coincident frequencies and stability of motion of a rigid panel in airflow are considered and discussed in detail.

2. Strong interaction

Consider an eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}, \tag{1}$$

where $\mathbf{A}$ is a real $m \times m$ non-symmetric matrix with the elements a_{ij} smoothly depending on a vector of real parameters $\mathbf{p} = (p_1, \ldots, p_n)$, λ is an eigenvalue, and $\mathbf{u}$ is a corresponding eigenvector. The eigenvalues λ are determined from the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where $\mathbf{I}$ is the identity matrix.

Assume that at $\mathbf{p}_0 = (p_1^0, \ldots, p_n^0)$ the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ possesses a double eigenvalue λ_0, as a root of the characteristic equation, with an eigenvector $\mathbf{u}_0$ and associated vector $\mathbf{u}_1$ satisfying the Jordan chain equations

$$\mathbf{A}_0\mathbf{u}_0 = \lambda_0\mathbf{u}_0,$$

$$\mathbf{A}_0\mathbf{u}_1 = \lambda_0\mathbf{u}_1 + \mathbf{u}_0. \tag{2}$$

Eqs. (2) mean that there is only one linearly independent eigenvector $\mathbf{u}_0$ corresponding to the double eigenvalue λ_0. Along with Eqs. (2), a left eigenvector $\mathbf{v}_0$ and associated vector $\mathbf{v}_1$ are defined by the left Jordan chain equations

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{A}_0 = \lambda_0\mathbf{v}_0^{\mathrm{T}},$$

$$\mathbf{v}_1^{\mathrm{T}}\mathbf{A}_0 = \lambda_0\mathbf{v}_1^{\mathrm{T}} + \mathbf{v}_0^{\mathrm{T}}. \tag{3}$$

The eigenvectors and associated vectors of problems (2) and (3) are related by the equalities [9]

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{u}_0 = 0, \quad \mathbf{v}_1^{\mathrm{T}}\mathbf{u}_0 = \mathbf{v}_0^{\mathrm{T}}\mathbf{u}_1 \neq 0. \tag{4}$$

Note that the eigenvectors and associated vectors are not defined uniquely. Upon assuming that the vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ are given, the following normalization conditions are introduced,

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{u}_1 = 1, \quad \mathbf{v}_1^{\mathrm{T}}\mathbf{u}_1 = 0, \tag{5}$$

uniquely determining the vectors $\mathbf{v}_0$ and $\mathbf{v}_1$.

The aim of the paper is to study the behaviour of two eigenvalues λ, which are coincident and equal to λ_0 at $\mathbf{p}_0$, with a change of the vector of parameters $\mathbf{p}$ in the vicinity of the initial point $\mathbf{p}_0$. For this purpose, a variation $\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e}$ is assumed, where $\mathbf{e} = (e_1, \ldots, e_n)$ is a vector of variation with $\|\mathbf{e}\| = 1$, and $\varepsilon > 0$ is a small parameter. As a result, the matrix $\mathbf{A}$ takes the increment

$$\mathbf{A}(\mathbf{p} + \varepsilon\mathbf{e}) = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \varepsilon^2\mathbf{A}_2 + \cdots, \tag{6}$$

where

$$\mathbf{A}_1 = \sum_{j=1}^{n} \frac{\partial\mathbf{A}}{\partial p_j} e_j, \quad \mathbf{A}_2 = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2\mathbf{A}}{\partial p_j \partial p_k} e_j e_k, \tag{7}$$

with the derivatives taken at $\mathbf{p}_0$. Due to these variations the eigenvalue λ_0 and the corresponding eigenvector $\mathbf{u}_0$ take increments, which can be given in the form of expansions [9]

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + \cdots,$$

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon^{1/2}\mathbf{w}_1 + \varepsilon\mathbf{w}_2 + \varepsilon^{3/2}\mathbf{w}_3 + \cdots. \tag{8}$$

By substituting expressions (6) and (8) into eigenvalue problem (1), a chain of equations for the unknowns $\lambda_1, \lambda_2, \ldots$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots$ is obtained. Solving these equations yields [9,14]

$$\lambda_1 = \pm\sqrt{\mathbf{v}_0^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_0}, \quad \lambda_2 = (\mathbf{v}_0^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_1 + \mathbf{v}_1^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_0)/2. \tag{9}$$

For the sake of convenience, the following notation is introduced

$$a_j + \mathrm{i}b_j = \mathbf{v}_0^{\mathrm{T}}\frac{\partial\mathbf{A}}{\partial p_j}\mathbf{u}_0,$$

$$c_j + \mathrm{i}d_j = \frac{1}{2}\left(\mathbf{v}_0^{\mathrm{T}}\frac{\partial \mathbf{A}}{\partial p_j}\mathbf{u}_1 + \mathbf{v}_1^{\mathrm{T}}\frac{\partial \mathbf{A}}{\partial p_j}\mathbf{u}_0\right),$$

$$\Delta p_j = p_j - p_j^0 = \varepsilon e_j, \quad j = 1, 2, \ldots, n, \tag{10}$$

where a_j, b_j, c_j, d_j are real constants, and i is the imaginary unit. Then, using expressions (8)–(10) gives

$$\lambda = \lambda_0 \pm \sqrt{\sum\nolimits_{j=1}^{n} (a_j + \mathrm{i}b_j)\Delta p_j} + \sum_{j=1}^{n} (c_j + \mathrm{i}d_j)\Delta p_j + o(\varepsilon). \tag{11}$$

Eq. (11) describes the increments of two eigenvalues λ, when the parameters $p_1, \ldots, p_n$ are changed under the assumption that ε is small. Due to the condition $\|\mathbf{e}\| = 1$ one has

$$\|\Delta \mathbf{p}\| = \|\varepsilon \mathbf{e}\| = \varepsilon \ll 1. \tag{12}$$

Thus, inequality (12) implies that all the increments $\Delta p_1, \ldots, \Delta p_n$ are small for their absolute values.

From expression (11) it is seen that when only one parameter, say the parameter p_1, is changed while others remain unchanged $\Delta p_j = 0$, $j = 2, \ldots, n$, then the speed of interaction $\mathrm{d}\lambda/\mathrm{d}p_1$ is infinite at $p_1 = p_1^0$. Indeed, following Eq. (11) yields

$$\frac{\mathrm{d}\lambda}{\mathrm{d}p_1} = \pm\frac{1}{2}\sqrt{\frac{a_1 + \mathrm{i}b_1}{p_1 - p_1^0}} + O(1). \tag{13}$$

Since $a_1 + \mathrm{i}b_1$ is a complex number, which is generally non-zero, $\mathrm{d}\lambda/\mathrm{d}p_1 \to \infty$ as $p_1 \to p_1^0$.

2.1. *Real eigenvalue* λ_0

Consider the case of a real double eigenvalue λ_0. In this case the eigenvectors $\mathbf{u}_0$, $\mathbf{v}_0$ and associated vectors $\mathbf{u}_1$, $\mathbf{v}_1$ can be chosen real and, hence, the constants $b_j = d_j = 0$, $j = 1, \ldots, n$, in Eq. (11). Let us fix the increments $\Delta p_2, \ldots, \Delta p_n$ and consider behaviour of the interacting eigenvalues depending on the increment Δp_1. Then, formula (11) can be written in the form

$$\lambda = \lambda_0 + X + \mathrm{i}Y + o(\varepsilon), \tag{14}$$

where

$$X + \mathrm{i}Y = \pm\sqrt{a_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right)} + c_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right) - \frac{c_1\alpha}{a_1} + \beta, \tag{15}$$

and α and β are small real numbers

$$\alpha = \sum_{j=2}^{n} a_j \Delta p_j, \quad \beta = \sum_{j=2}^{n} c_j \Delta p_j. \tag{16}$$

Real quantities X and Y describe, respectively, the real and imaginary parts of the leading terms in eigenvalue perturbation (11). If $a_1(\Delta p_1 + \alpha/a_1) > 0$, then Eq. (15) yields

$$X = \pm\sqrt{a_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right)} + c_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right) - \frac{c_1\alpha}{a_1} + \beta, \quad Y = 0. \tag{17}$$

If $a_1(\Delta p_1 + \alpha/a_1) < 0$, then separating the real and imaginary parts in Eq. (15) gives

$$X = c_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right) - \frac{c_1\alpha}{a_1} + \beta,$$

$$Y = \pm\sqrt{-a_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right)}. \tag{18}$$

Eliminating $\Delta p_1 + \alpha/a_1$ from Eqs. (18) yields the parabola

$$X + \frac{c_1}{a_1}Y^2 = \beta - \frac{\alpha c_1}{a_1} \tag{19}$$

in the (X, Y) plane symmetric with respect to the X-axis. Since α and β are small numbers depending on Δp_j, $j = 2, \ldots, n$, parabola (19) gives trajectories of λ on the complex plane with a change of the parameter p_1 while the others remain fixed. Note that the constants a_1 and c_1 involved in Eq. (19) are taken at the initial point $\mathbf{p}_0$ in the parameter space.

First, assume that $a_1 > 0$ and $\Delta p_2 = \cdots = \Delta p_n = 0$, which implies $\alpha = \beta = 0$. It follows from Eqs. (17)–(19) that with the increase of Δp_1 the eigenvalues come together along parabola (19), merge to λ_0 at $\Delta p_1 = 0$, and then diverge along the real axis in opposite directions. The general picture of strong interaction is shown in Fig. 2, where $\xi = 0$ and the arrows indicate motion of the eigenvalues, when Δp_1 increases. The case $a_1 < 0$ implies the change of direction of motion for the eigenvalues.

If $\Delta p_2, \ldots, \Delta p_n$ are non-zero and fixed, then the constants α and β are generally non-zero. This means the shift of parabola (19) along the real axis by $\xi = \beta - \alpha c_1/a_1$; see Fig. 2. One can see that the double eigenvalue does not disappear. It changes to $\lambda_0 + \xi + o(\varepsilon)$ and appears at $p_1 = p_1^0 - \alpha/a_1 + o(\varepsilon)$.

2.2. Complex eigenvalue λ_0

Consider a complex eigenvalue λ_0. In this case the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{v}_0$, and $\mathbf{v}_1$ are complex and, hence, the constants b_j and d_j in expression (11) are generally non-zero. Keeping the lowest order term in Eq. (11) gives

$$\lambda = \lambda_0 + X + \mathrm{i}Y + o(\varepsilon^{1/2}), \tag{20}$$

where

$$X + \mathrm{i}Y = \pm\sqrt{\sum_{j=1}^{n}(a_j + \mathrm{i}b_j)\Delta p_j}. \tag{21}$$

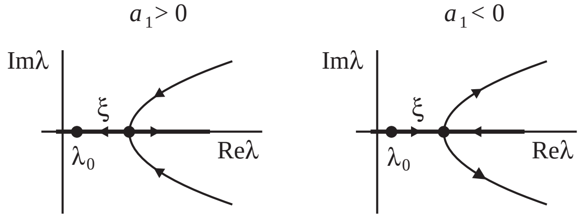

Fig. 2. Strong interaction of eigenvalues (real λ_0).

Taking the square of Eq. (21) and separating real and imaginary parts, one gets the equations

$$X^2 - Y^2 = \sum_{j=1}^{n} a_j \Delta p_j,$$

$$2XY = \sum_{j=1}^{n} b_j \Delta p_j. \tag{22}$$

Expressing the increment Δp_1 from one of equations (22) and substituting it into the other equation yields

$$X^2 - \frac{2a_1}{b_1} XY - Y^2 = \gamma, \tag{23}$$

where γ is a small real constant:

$$\gamma = \sum_{j=2}^{n} \left(a_j - \frac{a_1 b_j}{b_1} \right) \Delta p_j. \tag{24}$$

In Eq. (23) it is assumed that $b_1 \neq 0$, which is the non-degeneracy condition for the complex eigenvalue λ_0. Eq. (23) describes trajectories of the eigenvalues λ, when only the parameter Δp_1 is changed and the increments $\Delta p_2, \ldots, \Delta p_n$ are fixed.

If all $\Delta p_j = 0, j = 2, \ldots, n$, or if they are non-zero, but satisfy the equality $\gamma = 0$, then Eq. (23) yields two perpendicular lines

$$X - \left(\frac{a_1}{b_1} + \sqrt{1 + \frac{a_1^2}{b_1^2}} \right) Y = 0,$$

$$X - \left(\frac{a_1}{b_1} - \sqrt{1 + \frac{a_1^2}{b_1^2}} \right) Y = 0, \tag{25}$$

which intersect at the origin. Two eigenvalues $\lambda = \lambda_0 + X + \mathrm{i}Y + o(\varepsilon^{1/2})$ approach along one of lines (25), merge to λ_0 at $\Delta p_1 = 0$, and then diverge along another line (25), perpendicular to the line of approach; see Fig. 3, where the arrows show motion of λ with increasing Δp_1. Strong interaction in the three-dimensional space $(p_1, \mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda)$ is shown in Fig. 1 (left) with two identical parabolae lying in perpendicular planes.

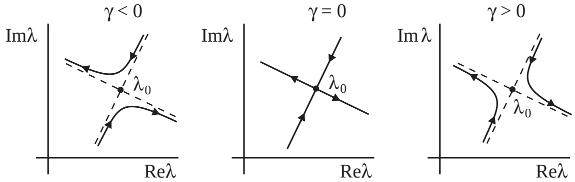

Fig. 3. Strong interaction of eigenvalues (complex λ_0).

If $\gamma \neq 0$, then Eq. (23) defines two hyperbolae in the (X,Y) plane with asymptotes (25). When Δp_1 increases, two eigenvalues come closer, turn, and diverge; see Fig. 3. When γ changes the sign, the quadrants containing hyperbola branches are changed to the adjacent.

Example 1. As an example, stability of vibrations of a rigid panel of infinite span in airflow is considered. It is assumed that the panel is maintained on two elastic supports with the stiffness coefficients k_1 and k_2 per unit span. The panel has two degrees of freedom: a vertical displacement y and a rotation angle φ; see Fig. 4. It is supposed that the aerodynamic lift force L is proportional to the angle of attack φ, the dynamic pressure of airflow, and the width b of the panel

$$L = c_y \frac{\rho v^2}{2} b\varphi. \tag{26}$$

Here, c_y is the aerodynamic coefficient, ρ and v are the density and speed of the flow respectively. It is assumed that m is the mass of the panel per unit surface. Damping forces are not involved in the model.

Small vibrations of the panel in airflow are described by the differential equations [2]

$$\ddot{y} + a_{11} y + a_{12}\varphi = 0,$$

$$\ddot{\varphi} + a_{21} y + a_{22}\varphi = 0, \tag{27}$$

where

$$a_{11} = \frac{k_1 + k_2}{mb}, \quad a_{12} = \frac{k_1 - k_2}{2m} - c_y \frac{\rho v^2}{2m},$$

$$a_{21} = \frac{6(k_1 - k_2)}{mb^2}, \quad a_{22} = \frac{3(k_1 + k_2)}{mb} - 3c_y \frac{\rho v^2}{2mb}. \tag{28}$$

It is convenient to introduce the non-dimensional variables

$$k = \frac{k_1 - k_2}{2(k_1 + k_2)}, \quad q = \frac{c_y \rho v^2}{2(k_1 + k_2)}, \quad \tilde{y} = \frac{y}{b}, \quad \tau = t\sqrt{\frac{k_1 + k_2}{mb}}, \tag{29}$$

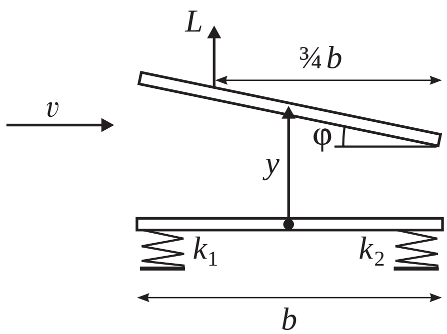

Fig. 4. A panel on elastic supports vibrating in airflow.

where k is a relative stiffness parameter changing in the interval $-1/2 \leqslant k \leqslant 1/2$, and $q \geqslant 0$ is a load parameter. Using these variables in Eqs. (27) and separating the time $(\tilde{y}, \varphi)^T = \mathbf{u}\exp(i\nu\tau)$, one obtains the eigenvalue problem (1) with

$$\mathbf{A} = \begin{pmatrix} 1 & k-q \\ 12k & 3-3q \end{pmatrix}, \quad \lambda = \nu^2. \tag{30}$$

The stability problem of motion of the panel depending on two parameters $\mathbf{p} = (q, k)$ has been studied in Ref. [15]. The characteristic equation is

$$\lambda^2 + (3q-4)\lambda + 12kq - 3q - 12k^2 + 3 = 0. \tag{31}$$

Motion of the panel is stable if all eigenvalues λ are positive and simple. The stability of the panel can be lost by divergence or by flutter, the boundaries of which are given by $\lambda = 0$ or double positive eigenvalues with a single eigenvector respectively. Setting the discriminant of Eq. (31) equal to zero gives

$$q_f = \frac{2}{3}\left(1 + 4k - 2\sqrt{k(k+2)}\right). \tag{32}$$

This is the boundary between flutter and stability regions; see Fig. 5, where S and F denote the stability and flutter regions respectively. It follows from Eq. (32) that the flutter region belongs to the half-plane $k \geqslant 0$. The other branch of the solution, with plus before the square root in Eq. (32), corresponds to the boundary between flutter and divergence regions [15], shown in Fig. 5 by dashed line.

Consider a point (k, q_f) on the flutter boundary (32). At this point, the eigenvalue problem (1) with Eqs. (30) and (32) is solved and the double eigenvalue $\lambda_0 = 2 - 3q_f/2$ with the corresponding eigenvectors and associated vectors satisfying normalization conditions (5) are found as

$$\mathbf{u}_0 = \begin{pmatrix} \dfrac{2(q_f - k)}{3q_f - 2} \\ 1 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} 0 \\ \dfrac{2}{2 - 3q_f} \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} 12k \\ \dfrac{2 - 3q_f}{2} \end{pmatrix},$$

$$\mathbf{v}_1 = \begin{pmatrix} \dfrac{24k}{3q_f - 2} \\ 0 \end{pmatrix}.$$

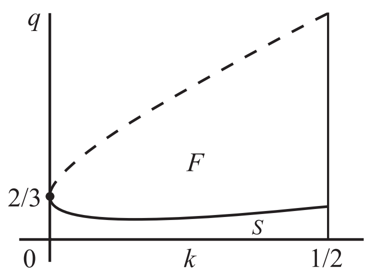

Fig. 5. The stability and flutter regions for the panel vibrating in airflow.

Then, according to Eqs. (10) the quantities become

$$a_1 = -\frac{3}{2}(8k - 3q_f + 2), \quad a_2 = 12(2k - q_f), \quad c_1 = -\frac{3}{2}, \quad c_2 = 0,$$

$$b_1 = b_2 = 0, \quad d_1 = d_2 = 0, \tag{33}$$

and the approximate formula (11) for the eigenvalues takes the form

$$\lambda = \frac{4 - 3q_f}{2} \pm \sqrt{-\frac{3}{2}(8k - 3q_f + 2)\Delta q + 12(2k - q_f)\Delta k} - \frac{3}{2}\Delta q. \tag{34}$$

This formula coincides with that of obtained from characteristic equation (31) with first order Taylor expansion of the terms under and out of the square root. The equation of parabola (19) becomes

$$Y^2 + (8k - 3q_f + 2)X = 12(q_f - 2k)\Delta k. \tag{35}$$

Due to expressions (32) and (33) the term a_1 is negative for $0 \leqslant k \leqslant 1/2$. This means that for $\Delta k = 0$ with the increase of q in the vicinity of the flutter boundary two positive eigenvalues λ approach each other, merge to $\lambda_0 = 2 - 3q_f/2$, become complex conjugate (flutter) and diverge along parabola (35). If $\Delta k \neq 0$ is small and fixed, then there is a shift of the double eigenvalue by $\xi = -a_2 c_1 \Delta k/a_1 = 12(q_f - 2k)\Delta k/(8k - 3q_f + 2)$ and a shift of the parameter q at which the eigenvalue becomes double (flutter boundary)

$$q_f(k + \Delta k) \approx q_f(k) - a_2 \Delta k/a_1 = q_f(k) + \frac{8(2k - q_f)}{8k - 3q_f + 2}\Delta k. \tag{36}$$

Note that these shifts are negative or positive depending on the sign of $2k - q_f$, which is negative for $0 \leqslant k < 2/\sqrt{3} - 1$ and positive for $2/\sqrt{3} - 1 < k \leqslant 1/2$. At $k = 2/\sqrt{3} - 1$ the function $q_f(k)$ takes the minimum; see Fig. 5.

3. Weak interaction

Consider a double eigenvalue λ_0 of the matrix $\mathbf{A}_0$ with two linearly independent eigenvectors. Such an eigenvalue is called semi-simple. Two right eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ and two left eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ are determined by the equations

$$\mathbf{A}_0\mathbf{u}_i = \lambda_0\mathbf{u}_i, \quad \mathbf{v}_i^{\mathrm{T}}\mathbf{A}_0 = \lambda_0\mathbf{v}_i^{\mathrm{T}}, \quad i = 1, 2,$$

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{u}_j = \delta_{ij}, \quad i,j = 1, 2, \tag{37}$$

where δ_{ij} is the Kronecker delta. The last equation represents the normalization condition determining left eigenvectors uniquely for given right eigenvectors. Note that any linear combination of right (or left) eigenvectors is a right (or left) eigenvector.

Consider a perturbation of the parameter vector $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$, where $\Delta\mathbf{p} = \varepsilon\mathbf{e}$ with a direction $\mathbf{e}$ in the parameter space ($\|\mathbf{e}\| = 1$) and a small positive perturbation parameter ε. Then, the eigenvalue λ_0 and corresponding eigenvector $\mathbf{u}_0$ take increments, which can be given in the form of expansions [9,13]

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots,$$

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon\mathbf{w}_1 + \varepsilon^2\mathbf{w}_2 + \cdots. \tag{38}$$

By substituting expressions (6) and (38) into eigenvalue problem (1), a chain of equations for the unknowns $\lambda_1, \lambda_2, \ldots$ and $\mathbf{u}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$ is obtained. Solution of these equations yields the vector

$$\mathbf{u}_0 = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2, \tag{39}$$

where γ_1 and γ_2 are unknown coefficients determined by (see [13])

$$\begin{pmatrix} \mathbf{v}_1^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_1 & \mathbf{v}_1^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_2 \\ \mathbf{v}_2^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_1 & \mathbf{v}_2^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \tag{40}$$

A non-trivial solution γ_1, γ_2 of this equation exists if and only if λ_1 is an eigenvalue of the 2×2 matrix on the left side. Two eigenvalues λ_1 of this matrix and corresponding eigenvectors $(\gamma_1, \gamma_2)^{\mathrm{T}}$ determine leading terms in the expansions for two eigenvalues λ and corresponding eigenvectors $\mathbf{u}$ (38), which appear due to bifurcation of the double semi-simple eigenvalue λ_0.

Introducing the notation

$$X + \mathrm{i}\,Y = \varepsilon\lambda_1, \tag{41}$$

where X and Y are, respectively, real and imaginary parts of the term $\varepsilon\lambda_1$, expansion for eigenvalue (38) can be written in the form

$$\lambda = \lambda_0 + X + \mathrm{i}\,Y + o(\varepsilon). \tag{42}$$

According to relations (7) and (40), $X + \mathrm{i}\,Y$ is an eigenvalue of the 2×2 matrix

$$\sum_{j=1}^{n} \begin{pmatrix} f_j^{11} & f_j^{12} \\ f_j^{21} & f_j^{22} \end{pmatrix} \Delta p_j, \tag{43}$$

where

$$f_j^{kl} = \mathbf{v}_k^{\mathrm{T}} \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_l, \tag{44}$$

with the derivatives evaluated at $\mathbf{p}_0$. Solving the characteristic equation for matrix (43) gives

$$X + \mathrm{i}\,Y = \sum_{j=1}^{n} g_j \Delta p_j \pm \sqrt{\sum\nolimits_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k}, \tag{45}$$

where

$$g_j = (f_j^{11} + f_j^{22})/2, \quad h_{jk} = (f_j^{11} - f_j^{22})(f_k^{11} - f_k^{22})/4 + (f_j^{12} f_k^{21} + f_j^{21} f_k^{12})/2. \tag{46}$$

Note that $h_{jk} = h_{kj}$ for any j and k. Expression (45) determines approximation of eigenvalues (42), when the parameter vector $\Delta\mathbf{p}$ is changing under the assumption that it is small for the absolute value. The coefficients g_j and h_{jk} in this expression depend on the left and right eigenvectors, corresponding to the eigenvalue λ_0, and first derivatives of the matrix $\mathbf{A}$ with respect to parameters taken at $\mathbf{p}_0$.

3.1. Real eigenvalue λ_0

Consider a real semi-simple eigenvalue λ_0. In this case the eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ can be chosen real and, hence, the coefficients f_j^{kl}, g_j, and h_{jk} in expressions (44) and (46) are real. By expressing the square root from equality (45) and taking the square of the obtained relation, two equations for real and imaginary parts are found as follows:

$$\left(X - \sum_{j=1}^{n} g_j \Delta p_j\right)^2 - Y^2 = \sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k,$$

$$2\left(X - \sum_{j=1}^{n} g_j \Delta p_j\right) Y = 0. \tag{47}$$

The second equation requires that $X = \sum_{j=1}^{n} g_j \Delta p_j$ or $Y = 0$. Therefore, two independent systems are obtained:

$$\left(X - \sum_{j=1}^{n} g_j \Delta p_j\right)^2 = \sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k, \quad Y = 0, \tag{48}$$

and

$$Y^2 = -\sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k, \quad X = \sum_{j=1}^{n} g_j \Delta p_j. \tag{49}$$

Consider now the behaviour of eigenvalues depending on the parameter p_1, when other parameters $p_2, \ldots, p_n$ are fixed. First, let us put the increments $\Delta p_2 = \cdots = \Delta p_n = 0$. Then, Eqs. (48) and (49) take the form

$$(X - g_1 \Delta p_1)^2 = h_{11} \Delta p_1^2, \quad Y = 0, \tag{50}$$

$$Y^2 = -h_{11} \Delta p_1^2, \quad X = g_1 \Delta p_1. \tag{51}$$

Upon assuming that $h_{11} \neq 0$ (the non-degenerate case), only one of systems (50) and (51) has non-zero solutions. If $h_{11} > 0$, then system (51) has only the zero solution $X = Y = \Delta p_1 = 0$, and system (50) yields

$$X = g_1 \Delta p_1 \pm \sqrt{h_{11}} \Delta p_1, \quad Y = 0. \tag{52}$$

Expression (52) describe two real eigenvalues (42), which cross each other at the point λ_0 on the complex plane as Δp_1 changes from negative to positive values; see Fig. 6 (r'), where the arrows show motion of the eigenvalues with an increment of Δp_1. If $h_{11} < 0$, then system (50) has only the zero solution, and system (51) yields

$$X = g_1 \Delta p_1, \quad Y = \pm \sqrt{-h_{11}} \Delta p_1. \tag{53}$$

These formulae describe two complex conjugate eigenvalues crossing at the point λ_0 on the real axis with a change of Δp_1; see Fig. 6 (r'').

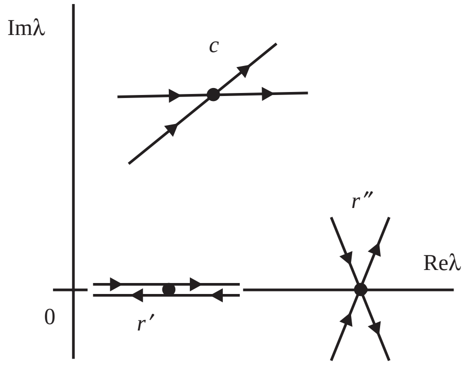

Fig. 6. Weak interaction of eigenvalues for $\Delta p_2 = \cdots = \Delta p_n = 0$.

From expressions (52) and (53) one can see that the weak interaction occurs at the same plane in the three-dimensional space $(p_1, \mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda)$, and the speed of interaction $\mathrm{d}\lambda/\mathrm{d}p_1$ remains finite, Fig. 1 (right).

Now, consider the case, when the increments $\Delta p_2, \ldots, \Delta p_n$ are small and fixed. Then, upon using the notation

$$\sum_{j=1}^{n} g_j \Delta p_j = g_1 \Delta p_1 + \sigma, \quad \sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k = h_{11}(\Delta p_1 - \delta)^2 + \psi, \tag{54}$$

where σ, δ, and ψ are small real constants depending on $\Delta p_2, \ldots, \Delta p_n$,

$$\sigma = \sum_{j=2}^{n} g_j \Delta p_j, \quad \delta = -\sum_{j=2}^{n} \frac{h_{1j}}{h_{11}} \Delta p_j, \quad \psi = \sum_{j,k=2}^{n} h_{jk} \Delta p_j \Delta p_k - h_{11}\delta^2, \tag{55}$$

Eqs. (48) and (49) take the form

$$(X - \sigma - g_1 \Delta p_1)^2 - h_{11}(\Delta p_1 - \delta)^2 = \psi, \quad Y = 0, \tag{56}$$

$$Y^2 + h_{11}(\Delta p_1 - \delta)^2 = -\psi, \quad X = \sigma + g_1 \Delta p_1. \tag{57}$$

Solutions of systems (56) and (57) depend qualitatively on the signs of the constants h_{11} and ψ. Under the non-degeneracy conditions $h_{11} \neq 0$ and $\psi \neq 0$ there are four possibilities.

Case r'_+ ($h_{11} > 0$, $\psi > 0$): System (56) determines two hyperbolae in the $(\Delta p_1, X)$ plane; system (57) has no solutions; see Fig. 7. Two simple real eigenvalues approach, and then diverge as Δp_1 is changed; a double eigenvalue does not appear; see Fig. 8.

Case r'_- ($h_{11} > 0$, $\psi < 0$): System (56) determines two hyperbolae in the $(\Delta p_1, X)$ plane; system (57) defines an ellipse in the $(\Delta p_1, Y)$ plane; see Fig. 7. The hyperbolae and ellipse have two common points

$$\Delta p_1^{\pm} = \delta \pm \sqrt{-\psi/h_{11}}, \quad X^{\pm} = \sigma + g_1 \Delta p_1^{\pm}, \quad Y^{\pm} = 0. \tag{58}$$

With increasing Δp_1 two simple real eigenvalues approach, interact strongly at $\Delta p_1^- = \delta - \sqrt{-\psi/h_{11}}$, become complex conjugate, interact strongly again at $\Delta p_1^+ = \delta + \sqrt{-\psi/h_{11}}$, and then diverge along the real axis; see Fig. 8. By eliminating Δp_1 from Eq. (57), an ellipse on the

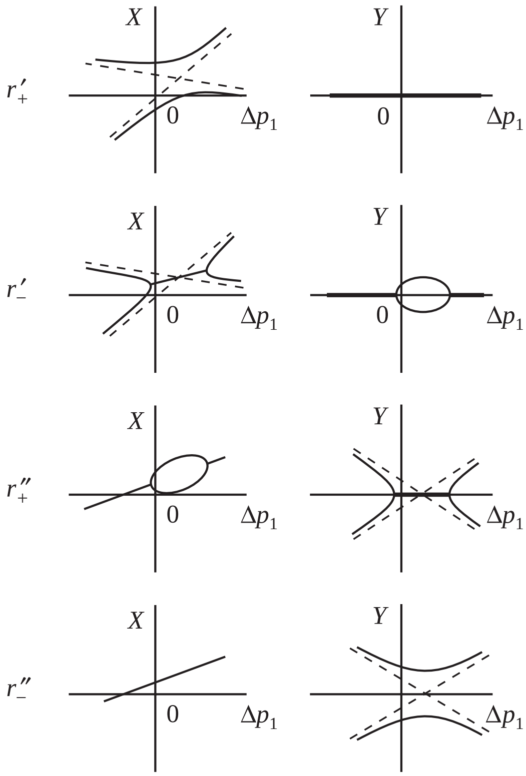

Fig. 7. Weak interaction of eigenvalues for small $\Delta p_2, \ldots, \Delta p_n$.

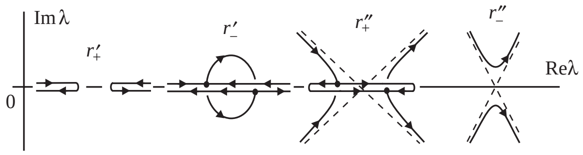

Fig. 8. Weak interaction of eigenvalues on the complex plane for small $\Delta p_2, \ldots, \Delta p_n$.

complex plane is obtained as

$$Y^2 + h_{11}\left(\frac{X - \sigma - g_1\delta}{g_1}\right)^2 = -\psi; \tag{59}$$

see Fig. 8.

If the eigenvalues are plotted in the $(p_1, \mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda)$ space, one observes a small elliptic bubble appearing from the point $(p_1^0, \lambda_0, 0)$; see Fig. 9. This bubble is placed in the plane perpendicular to the plane of original interaction.

At points (58) the double real eigenvalues

$$\lambda^{\pm} = \lambda_0 + X^{\pm} + o(\varepsilon), \tag{60}$$

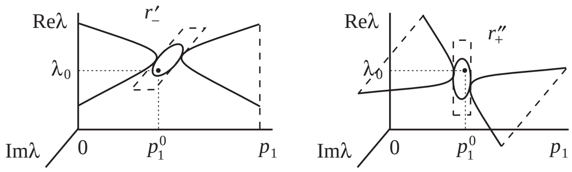

Fig. 9. Weak interaction of eigenvalues in the $(p_1, \mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda)$ space for small $\Delta p_2, \ldots, \Delta p_n$ in the cases r'_- and r''_+.

appear. It is easy to show that each of these eigenvalues has a single eigenvector. Indeed, if eigenvalues (60) were semi-simple, then $X^\pm$ have to be semi-simple eigenvalues of 2×2 matrix (43) at $\Delta p_1 = \Delta p_1^\pm$. Hence, matrix (43) becomes

$$\begin{pmatrix} X^\pm + f_1^{11}(\Delta p_1 - \Delta p_1^\pm) & f_1^{12}(\Delta p_1 - \Delta p_1^\pm) \\ f_1^{21}(\Delta p_1 - \Delta p_1^\pm) & X^\pm + f_1^{22}(\Delta p_1 - \Delta p_1^\pm) \end{pmatrix}. \tag{61}$$

Using this matrix with expressions (46) and (54), gives

$$\begin{aligned} h_{11}(\Delta p_1 - \delta)^2 + \psi &= \sum_{j,k=1}^{n} h_{ij}\Delta p_j \Delta p_k \\ &= ((f_1^{11} - f_1^{22})^2/4 + f_1^{12}f_1^{21})(\Delta p_1 - \Delta p_1^\pm)^2 = h_{11}(\Delta p_1 - \Delta p_1^\pm)^2 \end{aligned} \tag{62}$$

and, hence, $\psi = 0$. But this is a contradiction to the assumption that $\psi < 0$. Therefore, two interactions at points (58) are strong and follow the scenarios described in the previous section.

Case r''_+ ($h_{11} < 0$, $\psi > 0$): System (56) determines an ellipse in the $(\Delta p_1, X)$ plane; system (57) defines two hyperbolae in the $(\Delta p_1, Y)$ plane; see Fig. 7. The hyperbolae and ellipse have two common points (58), where double real eigenvalues (60) appear and cause strong interactions of eigenvalues. Therefore, with a monotonous change of Δp_1 two complex conjugate eigenvalues approach, interact strongly at $\Delta p_{1^-} = \delta - \sqrt{-\psi/h_{11}}$, become real, interact again at $\Delta p_{1^+} = \delta + \sqrt{-\psi/h_{11}}$, then become complex conjugate and diverge; see Fig. 8. For this case Eq. (59) gives hyperbolae on the complex plane. The behaviour of eigenvalues in the three-dimensional space $(p_1, \mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda)$ is shown in Fig. 9, where one can see a small elliptic bubble appearing in the plane $\mathrm{Im}\,\lambda = 0$ perpendicular to the plane of original interaction.

Case r''_- ($h_{11} < 0$, $\psi < 0$): System (56) has no solutions; system (57) determines two hyperbolae in the $(\Delta p_1, Y)$ plane symmetric with respect to the Δp_1 axis; see Fig. 7. Two complex conjugate eigenvalues approach, and then diverge with an increment of Δp_1; a double eigenvalue does not appear; see Fig. 8. Note that hyperbolae (59) change the vertical angles, where they appear, compared to the case r''_+.

Variations of the parameters $\Delta p_2, \ldots, \Delta p_n$ change a picture of weak interaction in two ways: either the double semi-simple real eigenvalue disappear and simple eigenvalues move along hyperbolae as Δp_1 changes, or the double semi-simple eigenvalue splits in two double eigenvalues with single eigenvectors, which leads to a couple of successive strong interactions with the appearance of a small bubble in the $(p_1, \mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda)$ space.

3.2. Complex eigenvalue λ_0

Finally, consider the case when a double semi-simple eigenvalue λ_0 is complex. In this case the eigenvalues $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ and, hence, the coefficients f_j^{kl}, g_j, and h_{jk} are complex. If $\Delta p_2 = \cdots = \Delta p_n = 0$, then expression (45) yields

$$X + \mathrm{i}Y = (g_1 \pm \sqrt{h_{11}})\Delta p_1, \tag{63}$$

where g_1 and h_{11} are complex numbers. With a change of Δp_1 two eigenvalues (42) cross each other at the point λ_0 on the complex plane; see Fig. 6 (c).

Upon assuming that the increments $\Delta p_2, \ldots, \Delta p_n$ are small and fixed, Eq. (45) yields

$$X + \mathrm{i}Y = \sigma + g_1\Delta p_1 \pm \sqrt{h_{11}}\sqrt{(\Delta p_1 - \delta)^2 + \psi/h_{11}}, \tag{64}$$

where σ, δ, and ψ are small complex numbers defined by expressions (55). Upon assuming that the second term under the square root in Eq. (64) is much smaller than the first term, the following formula can be deduced:

$$\begin{aligned} X + \mathrm{i}Y &\approx \sigma + g_1\delta + g_1(\Delta p_1 - \delta) \pm \sqrt{h_{11}}\left((\Delta p_1 - \delta) + \frac{\psi}{2h_{11}(\Delta p_1 - \delta)}\right) \\ &= \sigma + g_1\delta + (g_1 \pm \sqrt{h_{11}})(\Delta p_1 - \delta) + o(\Delta p_1 - \delta) \end{aligned} \tag{65}$$

showing that the main directions of eigenvalues on the complex plane before and after the weak interaction remain the same as for unperturbed case (63).

The expression under the square root in Eq. (64)

$$z = (\Delta p_1 - \delta)^2 + \psi/h_{11}, \tag{66}$$

defines a parabola in the complex plane with a change of Δp_1; see Fig. 10 (in the case $\mathrm{Im}\,\delta = 0$ the parabola degenerates to a ray). Computing points z_1 and z_2 of the parabola belonging to the imaginary axis, which is perpendicular to the axis of the parabola, results in

$$\eta = z_1 z_2 = 4(\mathrm{Im}\,\delta)^4 - 4(\mathrm{Im}\,\delta)^2\,\mathrm{Re}\,\frac{\psi}{h_{11}} - \left(\mathrm{Im}\,\frac{\psi}{h_{11}}\right)^2 \in \mathbb{R}. \tag{67}$$

Assume that $\eta \neq 0$, which is a non-degenerate case. This means that $z \neq 0$ for all Δp_1 and, hence, two values of $X + \mathrm{i}Y$ given by expression (64) are different. As a result, eigenvalues (42) are different and the double eigenvalue disappears.

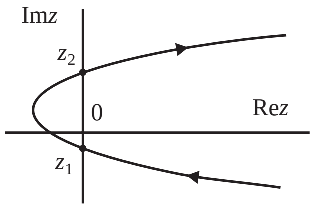

Fig. 10. Image of the function $z(\Delta p_1)$ with monotonic change of Δp_1.

If $\eta > 0$, then two purely imaginary points z_1 and z_2 lie at different sides of the origin, i.e., the origin belongs to the interior of the parabola. In this case z makes a turn around the origin as Δp_1 changes. This means that eigenvalues (42) approach, and then diverge without a change of direction as shown in Fig. 11 (c_+). If $\eta < 0$, then the origin lies outside the parabola (this condition remains valid, when the parabola does not intersect the imaginary axis). As a result, eigenvalues (42) approach, and then diverge with a change of direction as shown in Fig. 11 (c_-).

Variations of the parameters $\Delta p_2, \ldots, \Delta p_n$ destroy a double semi-simple complex eigenvalue. A picture of weak interaction can change in two ways: either eigenvalues follow the same directions after passing the neighbourhood of λ_0, or the eigenvalues interchange their directions. Behaviour of the eigenvalues in the neighbourhood of λ_0 can be rather complicated due to the square root of complex expression in formula (64).

Example 2. Consider a linear conservative system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\mathbf{x} = 0, \tag{68}$$

where $\mathbf{x} \in \mathbb{R}^m$ is a vector of generalized coordinates; $\mathbf{M}$ and $\mathbf{C}$ are symmetric positive definite real matrices of size $m \times m$ smoothly depending on a vector of two real parameters $\mathbf{p} = (p_1, p_2)$. Taking a solution of this system in the form $\mathbf{x} = \mathbf{u}\exp(i\omega t)$, the following eigenvalue problem is obtained,

$$\mathbf{C}\mathbf{u} = \omega^2\mathbf{M}\mathbf{u}, \tag{69}$$

where $\omega \in \mathbb{R}$ is a frequency and $\mathbf{u}$ is a mode of vibrations. Upon denoting

$$\mathbf{A} = \mathbf{M}^{-1}\mathbf{C}, \quad \lambda = \omega^2, \tag{70}$$

Eq. (69) can be written in standard form (1).

Consider now a point $\mathbf{p}_0$ in the parameter space, where the matrix $\mathbf{A}_0 = \mathbf{M}_0^{-1}\mathbf{C}_0$ has a double eigenvalue $\lambda_0 = \omega_0^2$. Since the matrices $\mathbf{M}_0$ and $\mathbf{C}_0$ are symmetric, the multiple eigenvalue λ_0 is always semi-simple. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be right eigenvectors (modes) corresponding to the eigenvalue λ_0. It can be shown that left eigenvectors are

$$\mathbf{v}_1 = \mathbf{M}_0\mathbf{u}_1, \quad \mathbf{v}_2 = \mathbf{M}_0\mathbf{u}_2. \tag{71}$$

Normalization conditions (37) take the form

$$\mathbf{u}_1^T\mathbf{M}_0\mathbf{u}_1 = \mathbf{u}_2^T\mathbf{M}_0\mathbf{u}_2 = 1, \quad \mathbf{u}_1^T\mathbf{M}_0\mathbf{u}_2 = \mathbf{u}_2^T\mathbf{M}_0\mathbf{u}_1 = 0. \tag{72}$$

Using expressions (70) and (71) in formula (44) gives

$$f_j^{kl} = \mathbf{u}_k^T\left(\frac{\partial\mathbf{C}}{\partial p_j} - \omega_0^2\frac{\partial\mathbf{M}}{\partial p_j}\right)\mathbf{u}_l, \tag{73}$$

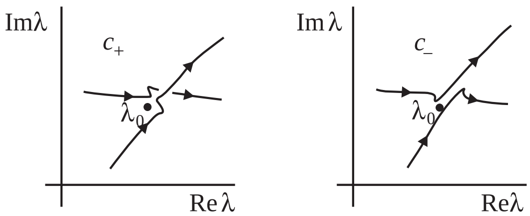

Fig. 11. Weak interaction of eigenvalues for small $\Delta p_2, \ldots, \Delta p_n$ (complex λ_0).

where $f_j^{12} = f_j^{21}$ due to the symmetry of the matrices **M** and **C**. With the use of relations (46), one obtains

$$g_1 = (f_1^{11} + f_1^{22})/2, \quad h_{11} = (f_1^{11} - f_1^{22})^2/4 + (f_1^{12})^2 \geqslant 0. \tag{74}$$

Upon assuming that $h_{11} \neq 0$, the bifurcation of the double eigenvalue λ_0 is given by expression (52) for the case $\Delta p_2 = 0$. This bifurcation is of the type r', see Fig. 6, where λ_0 splits into two real eigenvalues for small increments Δp_1. This agrees with the general theory, which says that all frequencies of the conservative system under consideration are real. If Δp_2 is non-zero and small, then expression (55) gives

$$\sigma = (f_2^{11} + f_2^{22})\Delta p_2/2,$$

$$\delta = -((f_1^{11} - f_1^{22})(f_2^{11} - f_2^{22})/4 + f_1^{12} f_2^{12})\Delta p_2/h_{11},$$

$$\psi = ((f_1^{11} - f_1^{22})f_2^{12} - (f_2^{11} - f_2^{22})f_1^{12})^2(\Delta p_2)^2/(4h_{11}) \geqslant 0. \tag{75}$$

Hence, behaviour of the eigenvalues with a change of Δp_1 is described by two hyperbolae (56); see Figs. 7 and 8 (r'_+). Two real eigenvalues approach, turn at some distance from each other, and diverge with a monotonic change of Δp_1. The frequencies

$$\omega = \sqrt{\lambda} = \sqrt{\omega_0^2 + X + \mathrm{i}Y + o(\varepsilon)} = \omega_0 + \frac{X + \mathrm{i}Y}{2\omega_0} + o(\varepsilon), \tag{76}$$

have the same type of behaviour in the neighbourhood of ω_0. A small perturbation of the second parameter Δp_2 destroys a picture of weak interaction in such a way that the double frequency disappears. This agrees with the results of singularity theory [21].

4. Conclusion

A new theory of interaction of eigenvalues in multi-parameter problems has been presented. This theory describes behaviour of eigenvalues with a change of parameters based on the information at the initial point, where the eigenvalues coincide. This information includes determination of the eigenvectors and associated vectors and first order derivatives of the system matrix with respect to parameters. The presented theory of interaction of eigenvalues in multi-parameter problems has a very broad field of applications because any physical system contains parameters. The study of behaviour of eigenvalues in vibrational systems is especially important for dynamic stability problems, since even a small change of parameters can lead to instability and catastrophic response of the system.

Interaction of two eigenvalues, which is obviously the most important case frequently taking place in vibrational problems, has been studied. The study of eigenvalues of higher multiplicity as well as investigation of some degenerate cases can be done using the methods developed in the present paper.

References

[1] V.V. Bolotin, Nonconservative Problems of the Theory of Elastic Stability, Pergamon, New York, 1963.

[2] Ya.G. Panovko, I.I. Gubanova, Stability and Oscillations of Elastic Systems, Consultants Bureau, New York, 1965.

[3] H. Ziegler, Principles of Structural Stability, Blaisdell, Waltham, MA, 1968.

[4] K. Huseyin, Vibrations and Stability of Multiple Parameter Systems, Sijthoff & Nordhoff, Alphen aan den Rijn, 1978.

[5] H. Leipholz, Stability Theory, B.G. Teubner, Stuttgart, 1987.

[6] J.M.T. Thompson, Instabilities and Catastrophes in Science and Engineering, Wiley, London, 1982.

[7] J.J. Thomsen, Vibrations and Stability. Order and Chaos, McGraw-Hill, London, 1997.

[8] M.P. Paidoussis, Fluid–Structure Interactions: Slender Structures and Axial Flow, Vol. 1, Academic Press, London, 1998.

[9] M.I. Vishik, L.A. Lyusternik, The solution of some perturbation problems for matrices and selfadjoint or non-selfadjoint differential equations, Russian Mathematical Surveys 15 (1960) 1–74.

[10] V.B. Lidskii, Perturbation theory of non-conjugate operators, USSR Computational Mathematics and Mathematical Physics 6 (1) (1966) 73–85.

[11] J. Moro, J.V. Burke, M.L. Overton, On the Lidskii–Vishik–Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure, SIAM Journal on Matrix Analysis and Applications 18 (4) (1997) 793–817.

[12] A.P. Seyranian, Sensitivity analysis of eigenvalues and development of instability, Strojnicky Casopis 42 (3) (1991) 193–208.

[13] A.P. Seyranian, Sensitivity analysis of multiple eigenvalues, Mechanics of Structures and Machines 21 (2) (1993) 261–284.

[14] A.A. Mailybaev, A.P. Seyranian, On singularities of a boundary of the stability domain, SIAM Journal on Matrix Analysis and Applications 21 (1) (1999) 106–128.

[15] A.P. Seyranian, O.N. Kirillov, Bifurcation diagrams and stability boundaries of circulatory systems, Theoretical and Applied Mechanics 26 (2001) 135–168.

[16] A.P. Seyranian, Collisions of eigenvalues in linear oscillatory systems, Journal of Applied Mathematics and Mechanics 58 (5) (1994) 805–813.

[17] A.P. Seyranian, P. Pedersen, On two effects in fluid/structure interaction theory, in: P.W. Bearman (Ed.), Flow Induced Vibration, A.A. Balkema, Rotterdam, 1995, pp. 565–576.

[18] O.N. Kirillov, A.P. Seyranian, Overlapping of frequency curves in nonconservative systems, Doklady Physics 46 (3) (2001) 184–189.

[19] A.A. Mailybaev, A.P. Seyranian, The stability domains of Hamiltonian systems, Journal of Applied Mathematics and Mechanics 63 (4) (1999) 545–555.

[20] A.P. Seyranian, W. Kliem, Bifurcations of eigenvalues of gyroscopic systems with parameters near stability boundaries, ASME Journal of Applied Mechanics 68 (2) (2001) 199–205.

[21] V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer, New York, 1978.