Chapter 8

Dual-Family Viscous Shock Waves in Systems of Conservation Laws: A Surprising Example *

Alexei A. Mailybaev
Institute of Mechanics, Moscow State Lomonosov University,
Michurinsky pr. 1, 119192 Moscow, Russia
mailybaev@imec.msu.ru

Dan Marchesin
Instituto Nacional de Matemática Pura e Aplicada – IMPA,
Estrada Dona Castorina, 110, 22460-320 Rio de Janeiro RJ, Brazil
marchesi@fluid.impa.br

Abstract: We consider shock waves satisfying the viscous profile criterion in a general system of conservation laws. We introduce a concept of $S_{i,j}$ dual-family shock wave, which is associated with a pair of characteristic families i and j. We develop a constructive method for sensitivity analysis of $S_{i,j}$ shocks. Generic solutions of the Riemann problem with $S_{i,j}$ shocks are described. As an example we present a system of three conservation laws. Remarkably, despite being coupled only through the viscous terms, it has an $S_{3,1}$ shock.

Keywords: Shock, viscous profile, conservation laws, sensitivity analysis, Riemann problem

Introduction

In this paper, shock waves in a general system of n conservation laws in one space dimension x are considered. When shock waves are required to possess viscous profiles rather than to satisfy Lax's inequalities, new types of shocks arise. In general, these shocks may be associated with the i-th characteristic family on the left and the j-th characteristic family on the right. We call such waves $S_{i,j}$ dual-family shocks. It was shown by [Kulikovskii, 1968] (see also [Kulikovskii et al., 2001]) that for $i > j$ the viscous profile requirement provides exactly the number of additional equations ($i - j$ equations) that is necessary to ensure that the number of characteristics emanating from the shock in positive time direction equals the number of independent conditions at the shock interface.

For systems of two equations, transitional shock waves ($i = j+1$) were studied by [Isaacson et al., 1990], [Schecter et al., 1996], and [Shearer et al., 1987], and novel structures of Riemann solutions resulting from such shocks were described. Shock waves with one or several additional equations for the viscous profile were found in problems of wave propagation in ferromagnetics, elastic media, and MHD, see [Kulikovskii et al., 2001], and in three phase flow in porous media they were analyzed for the case $S_{2,1}$ by [Plohr and Marchesin, 2001]. A program for studying the Hadamard stability of $S_{i,j}$ shocks was initiated in [Liu and Zumbrun, 1995], where an example of $S_{3,1}$ shock was presented.

In this paper, we provide a constructive method for perturbation analysis of general dual-family shocks, in which relationships between states at opposite sides of the shock and shock speed resulting from perturbations of problem parameters are derived. The role of $S_{i,j}$ shocks in generic solutions of the Riemann problem is described. As an example, we exhibit $S_{3,1}$ shocks in a particularly simple system of three conservation laws that are coupled only through viscous terms; five separated waves appear in the Riemann solution containing this $S_{3,1}$ shock.

8.1 Dual-family shock waves

We consider a system of partial differential equations of the form

$$\frac{\partial G(U)}{\partial t} + \frac{\partial F(U)}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D(U)\frac{\partial U}{\partial x}\right), \quad t \geq 0, \ x \in \mathbb{R} \qquad (8.1.1)$$

in the vanishing viscosity limit $\varepsilon \searrow 0$. The function representing conserved quantities $G(U) \in \mathbb{R}^n$, the flux function $F(U) \in \mathbb{R}^n$, and the $n \times n$ viscosity matrix $D(U)$ depend smoothly on the state vector $U \in \mathbb{R}^n$. Taking $\varepsilon = 0$ in (8.1.1) yields a system of n first-order conservation laws. Real eigenvalues $\lambda(U)$ of the characteristic

equation $\det(\partial F/\partial U - \lambda \, \partial G/\partial U) = 0$ are the characteristic speeds. When they are real and distinct in a region of state space U (the strictly hyperbolic region), we list them in increasing order $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

A shock wave is a discontinuous (weak) solution of system (8.1.1) with $\varepsilon = 0$ consisting of a left state $U_- = \lim_{x/t \nearrow s} U(x,t)$ and a right state $U_+ = \lim_{x/t \searrow s} U(x,t)$, where s is the shock speed. A shock wave is considered admissible if it agrees with the traveling wave solution (or viscous profile) $U(x,t) = U(\zeta)$, $\zeta = (x - st)/\varepsilon$ of system (8.1.1) in the vanishing viscosity limit $\varepsilon \searrow 0$. The admissibility condition implies that $U(\zeta)$ is a solution of the system of ordinary differential equations

$$D(U)\dot{U} = F(U) - F(U_-) - s(G(U) - G(U_-)), \tag{8.1.2}$$

"connecting" the left and right equilibria $U(-\infty) = U_-$, $U(+\infty) = U_+$.

By linearizing (8.1.2) about U_- and U_+ we obtain

$$\Delta\dot{U} = B(U_\pm, s)\Delta U, \qquad \Delta U(\zeta) = U(\zeta) - U_\pm, \tag{8.1.3}$$

where $B(U,s)$ is the $n \times n$ matrix

$$B(U, s) = \frac{\partial}{\partial U}\Big[D^{-1}(U)\big(F(U) - F(U_-) - s(G(U) - G(U_-))\big)\Big]. \tag{8.1.4}$$

Let $\mu_i(U, s)$, $i = 1, \ldots, n$, be the eigenvalues of $B(U, s)$ ordered with increasing real parts $\operatorname{Re}\mu_1 \le \operatorname{Re}\mu_2 \le \cdots \le \operatorname{Re}\mu_n$. Let us assume that

$$\operatorname{Re}\mu_{i-1}(U_-, s) < 0 < \operatorname{Re}\mu_i(U_-, s),$$
$$\operatorname{Re}\mu_j(U_+, s) < 0 < \operatorname{Re}\mu_{j+1}(U_+, s) \tag{8.1.5}$$

(if $i - 1 = 0$ or $j + 1 = n + 1$, the corresponding inequality is disregarded). One can see that $\mu_i(U_-, s) = 0$ and $\mu_j(U_+, s) = 0$ if $s = \lambda_i(U_-)$ and $s = \lambda_j(U_+)$, respectively. Under rather general conditions (e.g. [Kulikovskii et al., 2001] and [Majda and Pego, 1985]), inequalities (8.1.5) reduce to

$$\lambda_{i-1}(U_-) < s < \lambda_i(U_-), \tag{8.1.6}$$

$$\lambda_j(U_+) < s < \lambda_{j+1}(U_+). \tag{8.1.7}$$

Now we define an $S_{i,j}$ shock as an admissible shock satisfying inequalities (8.1.5). Shocks with $\mu_i(U_-, s) = 0$, $\mu_j(U_+, s) = 0$, and $\mu_i(U_-, s) = \mu_j(U_+, s) = 0$ are denoted by $S_{i,j}^-$, $S_{i,j}^+$, and $S_{i,j}^\pm$, respectively.

For $i = j$, inequalities (8.1.6) and (8.1.7) are the Lax conditions. Thus, an $S_{i,i}$ shock is a classical i-shock. Shocks with $i < j$ are called overcompressive. For

$i = j+1$ such a shock is termed transitional or undercompressive. We will focus on the case $i \geq j$, as according to [Isaacson et al., 1990], [Kulikovskii et al., 2001], and [Schecter et al., 1996], generically only for shocks with $i \geq j$ there may be a unique solution of the linearized problem for interaction with small perturbations.

Inequalities (8.1.6) coincide with the Lax conditions for the left state U_- of an i-shock. Analogously, inequalities (8.1.7) coincide with the Lax conditions for the right state U_+ of a j-shock. Therefore, the $S_{i,j}$ shock can be seen as a *dual-family shock wave* associated with the i-th characteristic family on the left and with the j-th characteristic family on the right. For the dual-family shocks $S_{i,j}^-$, $S_{i,j}^+$, and $S_{i,j}^\pm$, the shock speeds coincide with the characteristic speeds at one or both sides.

8.2   Sensitivity analysis of an $S_{i,j}$ shock

Let us consider $S_{i,j}$ as a point in the space (U_-, U_+, s) of the left and right states and speed of shocks. The set of all $S_{i,j}$ shocks generically defines a smooth surface $\mathcal{S}_{i,j}$ of dimension $n - i + j + 1$ in the space (U_-, U_+, s), which can be given locally by $n + i - j$ equations. Basic n equations relating U_-, U_+, and s are the Rankine–Hugoniot conditions

$$\mathcal{H}(U_-, U_+, s) \equiv F(U_+) - F(U_-) - s(G(U_+) - G(U_-)) = 0 \in \mathbb{R}^n, \qquad (8.2.1)$$

which follow from requiring that U_+ is an equilibrium of (8.1.2). For $i > j$, there are $i - j$ additional requirements

$$\mathcal{H}^{add}(U_-, U_+, s) = 0 \in \mathbb{R}^{i-j} \qquad (8.2.2)$$

determined by the existence of the viscous profile. Boundaries of the manifold $\mathcal{S}_{i,j}$ are typically related to characteristic shocks $S_{i,j}^-$, $S_{i,j}^+$, and $S_{i,j}^\pm$, or to bifurcations of the viscous profile, see [Schecter et al., 2001].

Let us consider a specific point $(U_-^0, U_+^0, s^0) \in \mathcal{S}_{i,j}$. A corresponding viscous profile $U^0(\zeta)$ satisfies equation (8.1.2) and the boundary conditions $U^0(-\infty) = U_-^0$, $U^0(+\infty) = U_+^0$. Linearizing equation (8.1.2) near the solution $U^0(\zeta)$, we obtain the system of ordinary differential equations

$$\dot{V} = B(U^0(\zeta), s^0)V, \quad V(-\infty) = V(+\infty) = 0, \quad V \in \mathbb{R}^n, \qquad (8.2.3)$$

where the matrix $B(U, s)$ is given in (8.1.4). The corresponding adjoint linear system takes the form

$$\dot{W} = -B^T(U^0(\zeta), s^0)W, \quad W(-\infty) = W(+\infty) = 0, \quad W \in \mathbb{R}^n. \qquad (8.2.4)$$

For any bounded function $X(\zeta) \in \mathbb{R}^n$, solutions $W(\zeta)$ of (8.2.4) have the property $\int_{-\infty}^{+\infty} W^T(\dot{X} - B(U^0(\zeta), s^0)X)d\zeta = 0$. Because of inequalities (8.1.5), the linear space $\mathcal{W}$ of solutions of (8.2.4) has dimension $\dim \mathcal{W} = i - j$; so we choose $i - j$ functions $W_1(\zeta), \ldots, W_{i-j}(\zeta)$ as a basis of $\mathcal{W}$.

In a neighborhood of the point (U_-^0, U_+^0, s^0), the manifold $\mathcal{S}_{i,j}$ is given by equations (8.2.1) and (8.2.2). The local form of $\mathcal{S}_{i,j}$ is determined by the linearization of these equations presented in the following theorem.

Theorem 1 *The tangent plane (dU_-, dU_+, ds) of the manifold $\mathcal{S}_{i,j}$ at the point $(U_-^0, U_+^0, s^0) \in \mathcal{S}_{i,j}$ is given by $d\mathcal{H} = 0$, $d\mathcal{H}^{add} = 0$ with*

$$d\mathcal{H} = \left(\frac{\partial F}{\partial U} - s^0\frac{\partial G}{\partial U}\right)_{U=U_+^0} dU_+ - \left(\frac{\partial F}{\partial U} - s^0\frac{\partial G}{\partial U}\right)_{U=U_-^0} dU_- \tag{8.2.5}$$
$$- (G_+^0 - G_-^0)ds,$$

$$d\mathcal{H}^{add} = \left(\int_{-\infty}^{+\infty} \hat{W}^T D_0^{-1}\left(\frac{\partial F}{\partial U} - s^0\frac{\partial G}{\partial U}\right)_{U=U_-^0} d\zeta\right) dU_-$$
$$+ \left(\int_{-\infty}^{+\infty} \hat{W}^T D_0^{-1}(G_0 - G_-^0)d\zeta\right) ds, \tag{8.2.6}$$

where $\hat{W}(\zeta) = [W_1(\zeta), \ldots, W_{i-j}(\zeta)]$ is an $n \times (i - j)$ matrix. The short notations $D_0(\zeta) = D(U^0(\zeta))$, $F_0(\zeta) = F(U^0(\zeta))$, $G_0(\zeta) = G(U^0(\zeta))$, $F_\pm^0 = F(U_\pm^0)$, and $G_\pm^0 = G(U_\pm^0)$ are used in (8.2.5), (8.2.6), where $U^0(\zeta)$ is the viscous profile of the $S_{i,j}$ shock at the initial point (U_-^0, U_+^0, s^0).

Equation (8.2.5) represents the differential of the Rankine–Hugoniot conditions (8.2.1), and (8.2.6) is the differential of the specially chosen equations (8.2.2). We omit the proof of the theorem, which is based on perturbation analysis of equation (8.1.2) using the adjoint linear system (8.2.4).

As a model of a physical system, equation (8.1.1) typically depends on one or several problem parameters. Under variations of these parameters, the functions $G(U)$, $F(U)$, and $D(U)$ undergo perturbations $\delta G(U)$, $\delta F(U)$, and $\delta D(U)$. If these perturbations are small, the manifold $S_{i,j}$ undergoes a small perturbation. The first order approximation of the perturbed manifold can be determined as follows.

Theorem 2 *Let $(U_-^0, U_+^0, s^0) \in \mathcal{S}_{i,j}$ and consider perturbations $\delta G(U)$, $\delta F(U)$, $\delta D(U)$ of the system functions. Then the first order approximation of the perturbed manifold $\mathcal{S}_{i,j}$ near the point (U_-^0, U_+^0, s^0) is given by the equations*

$$d\mathcal{H} = -\delta F_+^0 + \delta F_-^0 + s^0(\delta G_+^0 - \delta G_-^0), \tag{8.2.7}$$

$$d\mathcal{H}^{add} = - \int_{-\infty}^{+\infty} \hat{W}^T D_0^{-1} \delta D_0 \, D_0^{-1} \big(F_0 - F_-^0 - s^0 (G_0 - G_-^0) \big) d\zeta$$

$$+ \int_{-\infty}^{+\infty} \hat{W}^T D_0^{-1} \big(\delta F_0 - \delta F_-^0 - s^0 (\delta G_0 - \delta G_-^0) \big) d\zeta, \tag{8.2.8}$$

where the differentials $d\mathcal{H}$ and $d\mathcal{H}^{add}$ are given by (8.2.5), (8.2.6).

Theorems 1 and 2 determine all nearby $S_{i,j}$ shock waves, even when problem parameters are changed, using the information on a particular shock and its viscous profile. This method is useful for constructing Riemann solutions possessing $S_{i,j}$ shocks, continuation procedures, and parametric analysis.

The characteristic shock waves $S_{i,j}^-$, $S_{i,j}^+$, and $S_{i,j}^\pm$ can be studied in the same way. In addition to equations (8.2.1) and (8.2.2), one should use conditions ensuring that the shock speed is equal to the corresponding characteristic speed at one or both sides of the shock.

8.3   Dual-family shocks in Riemann solutions

The basic initial-value problem for a system of conservation laws (equations (8.1.1) with $\varepsilon = 0$) is the Riemann problem, given by piecewise constant initial data with a single jump at $x = 0$: $U(x,0) = U_L$ for $x < 0$ and $U(x,0) = U_R$ for $x > 0$. The solution is found in the form $U(x,t) = \hat{U}(\xi)$, $\xi = x/t$, consisting of continuously changing waves (rarefaction waves), jump discontinuities (shock waves), and separating constant states. Classically, there are n rarefactions, one for each characteristic speed, which we denote by $R_1, \ldots, R_n$. We require all shocks to have viscous profiles, i.e., there can be $S_{i,j}$, $S_{i,j}^-$, $S_{i,j}^+$, and $S_{i,j}^\pm$ shocks.

The structure of a Riemann solution is given by a sequence of waves w_k

$$w_1, \ w_2, \ \ldots, \ w_m, \tag{8.3.1}$$

appearing with increasing value of ξ. Here each wave $w_k \in \{R_i, S_{i,j}, S_{i,j}^-, S_{i,j}^+, S_{i,j}^\pm\}$ is a rarefaction or shock. The wave w_k has left and right states $U_{(k)-}$ and $U_{(k)+}$ and speeds $\xi_{(k)-} < \xi_{(k)+}$ for a rarefaction wave and $s_{(k)} = \xi_{(k)-} = \xi_{(k)+}$ for a shock wave. The left state of the first wave w_1 and the right state of the last wave w_m are the initial conditions of Riemann problem: $U_{(1)-} = U_L$ and $U_{(m)+} = U_R$. The natural requirements in sequence (8.3.1) are $U_{(k)+} = U_{(k+1)-}$ and $\xi_{(k)+} \le \xi_{(k+1)-}$. If $\xi_{(k)+} < \xi_{(k+1)-}$ then there is a separating constant state between w_k and w_{k+1}. In this case we will use the notation $w_k \mathbin{—} w_{k+1}$. If $\xi_{(k)+} = \xi_{(k+1)-}$ then the waves do not possess a separating constant state. This situation will be denoted by $w_k \,|\, w_{k+1}$.

The most important structures of a Riemann solution are the generic ones: they do not change under perturbations of initial conditions U_L, U_R and of system functions. Only shocks with $i \geq j$ may appear in generic structures; recall that the shock $S_{i,i}$ (or simply S_i) is a classical shock of i-th family. Overcompressive shocks ($i < j$) bifurcate to a set of waves under perturbations with arbitrarily small amplitudes. The following theorem describes all generic structures of a Riemann solution (see also [Schecter et al., 1996] for the case of two conservation laws).

Theorem 3 *If (8.3.1) is a generic structure of a Riemann solution, then* $w_1 \in \{R_1, S_1, S_1^+\}$, $w_m \in \{R_n, S_n, S_n^-\}$, *and each pair* w_k, w_{k+1} *has one of the types*

$$\{R_j \text{ or } S_{i,j}\} - \{R_{i'} \text{ or } S_{i',j'}\}, \quad i' = j + 1, \ i \geq j, \ i' \geq j',$$

$$R_i \ | \ \{S_{i,j}^- \text{ or } S_{i,j}^\pm\}, \quad i \geq j, \tag{8.3.2}$$

$$\{S_{i,j}^+ \text{ or } S_{i,j}^\pm\} \ | \ R_j, \quad i \geq j.$$

A classical Riemann solution consists of n separated waves R_i or S_i corresponding to different characteristic families. The classical structure $R_1 - R_2 - S_3$ of a Riemann solution in a system of three conservation laws is shown in Fig. 8.3.1(a) using characteristic lines in the space-time plane (shocks are shown as bold lines and rarefactions as thin line fans).

As an example, we list two nonclassical Riemann solution structures:

$$R_1 - S_2 - R_3 \,|\, S_{3,2}^- - R_3, \tag{8.3.3}$$

$$R_1 - S_2 - S_{3,1} - S_2 - R_3. \tag{8.3.4}$$

The Riemann solution with structure (8.3.3) contains a transitional characteristic shock $S_{3,2}^-$. A distinctive feature of structures (8.3.3) and (8.3.4) is that the classical waves R_3 and S_2 appears twice. Riemann solutions with these structures are shown in Fig. 8.3.1(b,c).

Riemann solutions with dual-family shock waves violate the classical structure of sequences of only n classical waves with increasing family number from left to right. The shocks with $i > j+1$ introduce a "jump back" capability in this sequence allowing classical waves of $(j+1), \ldots, (i-1)$-th characteristic families to appear repeatedly. Moreover, from the theoretical point of view, there is no general bound on the number of separated classical waves or of nonclassical shock waves in a Riemann solution for systems of $n > 2$ conservation laws. The existence of several separated waves with the same characteristic family is a property of Riemann solutions that has not appeared in previous works.
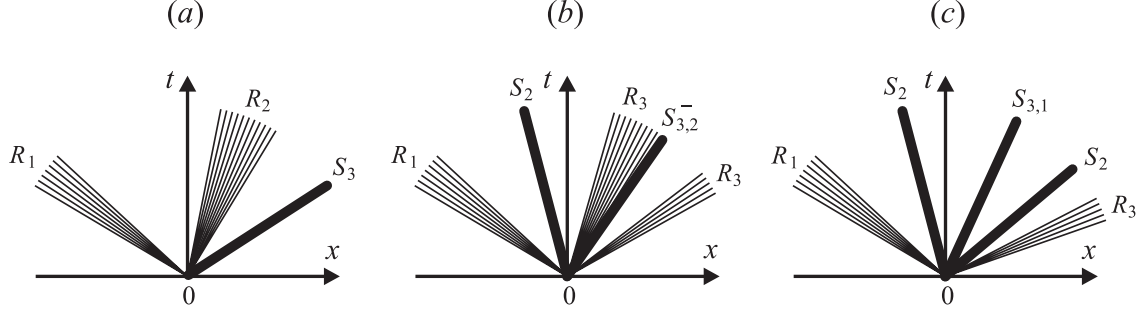
Figure 8.3.1: Riemann solutions: (a) classical, (b) with $S_{3,2}^-$ shock, (c) with $S_{3,1}$ shock.

8.4   Nonclassical shocks in three conservation laws coupled through the viscosity term

Let us consider system (8.1.1) with state vector $U = (u, v, w)^T$ and

$$G(U) = U, \quad F(U) = \begin{pmatrix} u^2 \\ v^2 \\ w^2 \end{pmatrix}, \quad D(U) = \begin{pmatrix} 9 & 8 & 2 \\ 8 & 9 & 2 \\ 2 & 2 & 1 \end{pmatrix}. \tag{8.4.1}$$

Equations in this system are coupled only through the viscous term $D\partial^2 U/\partial x^2$ with positive definite matrix D. Putting $\varepsilon = 0$ yields three uncoupled inviscid Burgers equations. Thus, classically this system has Riemann solutions containing three waves of different families.

Consider a shock with states $U_-^0 = (-3, 7, -1)^T$, $U_+^0 = (5, -5, 3)^T$ and speed $s^0 = 2$. The choice of D in (8.4.1) facilitates the verification that this shock is of type $S_{3,1}$ with straight line viscous profile

$$U^0(\zeta) = (1, 1, 1)^T + \rho(\zeta)(4, -6, 2)^T, \quad \rho(\zeta) = \tanh(2\zeta). \tag{8.4.2}$$

Two independent solutions of adjoint system (8.2.4) are

$$W_1(\zeta) = (-1, 0, 2)^T \exp\left(-8 \int_0^\zeta \rho(\zeta')d\zeta'\right),$$

$$W_2(\zeta) = (12, 9, 3)^T \exp\left(-\tfrac{4}{3} \int_0^\zeta \rho(\zeta')d\zeta'\right). \tag{8.4.3}$$

Using Theorem 1, we find the approximation of the manifold $\mathcal{S}_{3,1}$ consisting of all possible states and speeds (U_-, U_+, s) in the neighborhood of the point (U_-^0, U_+^0, s^0), in the form (dU_-, dU_+, ds) satisfying

$$dU_- + dU_+ = (1, 1, 1)^T ds, \quad \begin{pmatrix} -2 & 0 & 4 \\ 4 & 3 & 1 \end{pmatrix} dU_- = \begin{pmatrix} 1 \\ 4 \end{pmatrix} ds. \tag{8.4.4}$$

Table 8.4.1: Riemann solution with the $S_{3,1}$ dual-family shock wave for the initial conditions $U_L^0 = (-6, 7, 1)^T$ and $U_R^0 = (-1, -5, 0)^T$.

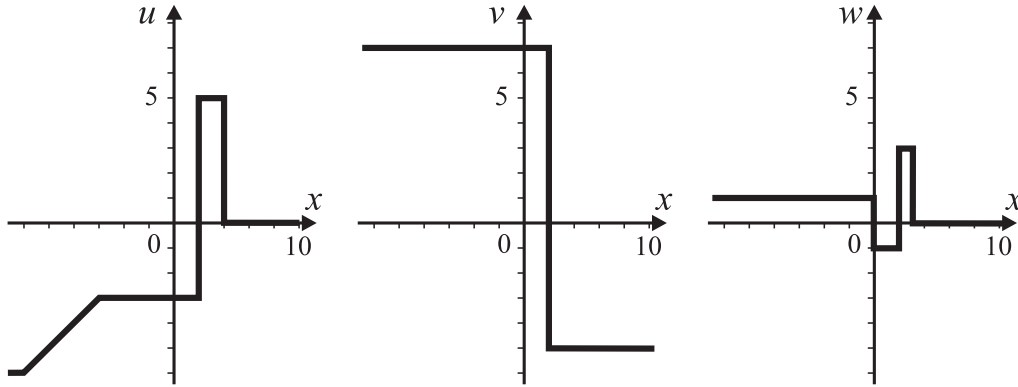| wave types	states	speeds
R_1	$U = (\xi/2, 7, 1)^T$	$-12 \leq \xi \leq -6$
S_2	$U_- = (-3, 7, 1)^T$, $U_+ = (-3, 7, -1)^T$	$s = 0$
$S_{3,1}$	$U_- = (-3, 7, -1)^T$, $U_+ = (5, -5, 3)^T$	$s = 2$
S_2	$U_- = (5, -5, 3)^T$, $U_+ = (5, -5, 0)^T$	$s = 3$
S_3	$U_- = (5, -5, 0)^T$, $U_+ = (-1, -5, 0)^T$	$s = 4$

Figure 8.4.2: Riemann solutions with $S_{3,1}$ shock wave.

It turns out that the approximation (8.4.4) coincides locally with the manifold $\mathcal{S}_{3,1}$, which is a part of plane (8.4.4) in the space (U_-, U_+, s).

The flux function variation $\delta F(U) = \delta p\,(v, w, u)^T$ (where p is a parameter) couples the conservation laws. Theorem 2 gives the perturbed manifold $\mathcal{S}_{3,1}$ near (U_-^0, U_+^0, s^0) for small values of δp as

$$dU_- + dU_+ = ds\,(1, 1, 1)^T + \delta p\,(3/2, 1/3, -2)^T,$$

$$\begin{pmatrix} -2 & 0 & 4 \\ 4 & 3 & 1 \end{pmatrix} dU_- = \begin{pmatrix} ds - 11\delta p/2 \\ 4ds + 5\delta p/2 \end{pmatrix}. \tag{8.4.5}$$

In Table 8.4.1 we give an example of a Riemann solution with the generic structure $R_1 - S_2 - S_{3,1} - S_2 - S_3$ containing an $S_{3,1}$ shock. Fig. 8.4.2 shows
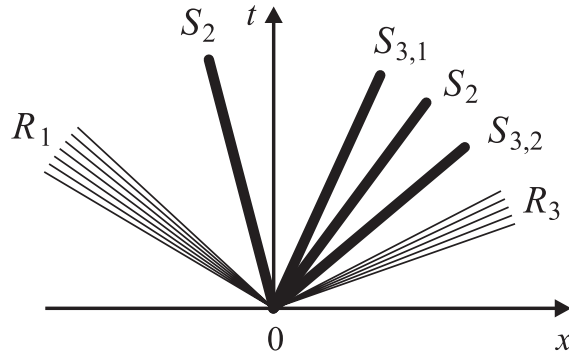
Figure 8.4.3: Riemann solution for a system of three equations with $S_{3,1}$ and $S_{3,2}$ dual-family shocks.

the space distribution for waves in this solution at $t = 1$. For any initial conditions (U_L, U_R) near (U_L^0, U_R^0) the solutions are similar.

Numerical experiments using a linearized Crank-Nicolson scheme were carried out for system (8.1.1), (8.4.1) with different initial conditions. It turned out that the solution of Table 8.4.1 is the only stable asymptotic solution for the given U_L^0 and U_R^0. The classical solution, obtained by solving each conservation law separately, consists of a rarefaction for the first state coordinate u and two shocks for v and w separated by constant states. Numerical calculations showed that this solution is unstable and that the shock S_3 corresponding to the change of v in the classical solution does not possess a viscous profile. If the right initial condition is changed to $U_R^0 = (-1, -5, 6)^T$, then the stable Riemann solution is even more complex: it contains six waves separated by constant states with structure R_1 — S_2 — $S_{3,1}$ — S_2 — $S_{3,2}$ — R_3; see Fig. 8.4.3.

These examples highlight the importance of dual-family shocks for the global structure of Riemann solutions of systems of conservation laws.

References

Isaacson, E.L., Marchesin, D. and Plohr, B.J. (1990). Transitional waves for conservation laws, *SIAM J. Math. Anal.* **21**, 837–866.

Kulikovskij, A.G. (1968). Surfaces of discontinuity separating two perfect media of different properties. Recombination waves in magnetohydrodynamics, *Prikl. Mat. Mekh.* **32**, 1125–1131 (in Russian).

Kulikovskii, A.G., Pogorelov, N.V. and Semenov, A.Yu. (2001). *Mathematical aspects*

of numerical solution of hyperbolic systems. Boca Raton: Chapman & Hall/CRC.

Liu, T.-P. and Zumbrun, K. (1995). On nonlinear stability of general undercompressive viscous shock waves, *Comm. Math. Phys.* **174**, 319–345.

Majda, A. and Pego, R.L. (1985). Stable viscosity matrices for systems of conservation laws, *J. Differential Equations* **56**, 229–262.

Plohr, B. and Marchesin, D. (2001). Wave structure in WAG recovery, SPE 71314, *Society of Petroleum Engineers Journal* **6**, 209–219.

Schecter, S., Marchesin, D. and Plohr, B.J. (1996). Structurally stable Riemann solutions, *J. Differential Equations* **126**, 303–354.

Schecter, S., Plohr, B.J. and Marchesin, D. (2001). Classification of codimension-one Riemann solutions, *J. Dynam. Differential Equations* **13**, 523–588.

Shearer, M., Schaeffer, D.G., Marchesin, D. and Paes-Leme, P.L. (1987). Solution of the Riemann problem for a prototype 2×2 system of nonstrictly hyperbolic conservation laws, *Arch. Rational Mech. Anal.* **97**, 299–320.