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MECHANICS

The Effect of Nonconservative Forces on the Stability of Systems with Multiple Frequencies and the Nicolai Paradox

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In 1928, E.L. Nicolai considered for the first time the problem on the stability of an elastic system loaded with follower forces [1]. In this work, Nicolai formulated the problem on the stability of the straight form of equilibrium of an elastic rod, one end of which is clamped, and the other end of which is loaded with a compressing force and a twisting follower (tangential) moment. Nicolai showed that the rod has no static forms of loss of stability in the case of identical principal moments of inertia of the cross section; however, it loses the stability dynamically for arbitrary small values of the twisting moment. This effect is known as the Nicolai paradox. In [1, 2], Nicolai showed that the presence of a small dissipation or the inequality of the principal moments of inertia of the rod cross section results in the system stabilization. These two important works of Nicolai were included in the collection of selected works [3]. It should be noted that Nicolai used a discrete model with two degrees of freedom for investigating the instability of the trivial solution and the analysis of the rod stabilization. For more on the Nicolai paradox and the development of further investigations in this field, see [4].

In our work, we formulate the problem on the instability of the conservative system with an arbitrary number of degrees of freedom with the addition of small nonconservative forces (positional and dissipative). Instability can arise when the conservative system has a multiple frequency of vibrations. The instability region was obtained in terms of matrices of perturbations and eigenvectors corresponding to the double frequency. In the absence of the dissipation, this region is bounded by a conical surface. The obtained results are generalized to the case of the frequency of an arbitrary multiplicity. In particular, it is shown that the addition of arbitrarily small circulatory forces typically results in the destabilization of the system with a multiple frequency. Then alongside with nonconservative positional forces, we investigate the effect of small dissipative forces. It is shown that, in this case, the addition of an arbitrarily small dissipation generally transforms the instability region from a cone into a two-sheeted hyperboloid increasing it discontinuously (the destabilization effect). Only for certain special ratios between the damping coefficients, the system is stabilized by small dissipative forces.

As a continuum analogue of the potential system with small nonconservative and dissipative forces, the Nicolai problem about the elastic stability of the compressed and twisted cantilever rod is considered, the applied twisting moment being assumed as tangential. The general formula for the instability region is obtained with taking into account a small distinction in principal moments of inertia of the rod cross section and small internal and external dissipative forces. For describing the internal friction, we use the Kelvin–Voigt viscoelastic model. We investigate the dependence of the instability region on the axial force. It proved to be that the obtained formula for the instability region is valid if we replace the tangential twisting moment for the axial one. This fact follows from the adjoint relation between the corresponding problems on eigenvalues.

PERTURBATION OF THE POTENTIAL SYSTEM WITH MULTIPLE FREQUENCIES

A linear vibrational system with nonconservative positional forces is described by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0, \quad \mathbf{C} = \mathbf{P} + \mathbf{N}, \tag{1}$$

where $\mathbf{q}$ is the vector of the generalized coordinates of an arbitrary dimension, $\mathbf{M}$ is the real symmetric positive definite mass matrix, and the real matrices $\mathbf{P} = \mathbf{P}^T$

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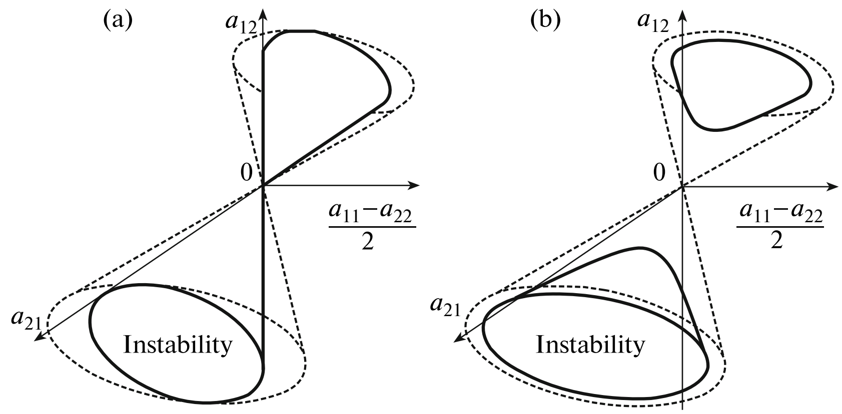

Fig. 1. Conical singularity of the stability region in the case of the perturbation of the system with a multiple frequency by small nonconservative forces: (a) without damping, and (b) with damping. The dashed conical surface designates the stability-region boundary in the case of infinitesimal damping.

and $\mathbf{N} = -\mathbf{N}^T$ describe the potential and nonconservative positional (circulatory) forces, respectively. Equation (1) has the solutions in the form of $\mathbf{q} = \mathbf{u}\exp(\lambda t)$, where the eigenvalues λ and the eigenvectors $\mathbf{u}$ are found from the eigenvalue problem

$$\mathbf{Cu} = \mu\mathbf{Mu}, \quad \mu = -\lambda^2. \tag{2}$$

It is obvious that alongside with λ, the values of $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$ are also eigenvalues, which reflects the property of reversibility of the system (1) in time. Hence, system (1) is stable if all eigenvalues λ are pure imaginary and simple or semisimple (the number of linearly independent eigenvectors $\mathbf{u}$ coincides with the algebraic multiplicity of λ); see, for example, [5]. This stability condition is equivalent to the statement that the eigenvalues μ in Eq. (2) are real, positive, and simple or semisimple. It should be noted that the simple eigenvalues λ cannot leave the imaginary axis due to a small perturbation of matrices of the system. Otherwise, it would mean also the occurrence of the eigenvalues $-\bar{\lambda}$, i.e., an increase in the total number of eigenvalues. Therefore, the stability-region boundary is characterized by multiple zero or multiple purely imaginary λ (zero or multiple positive μ).

In this work, we consider the case when the matrices of the system have the form of

$$\mathbf{M} = \mathbf{M}_0 + \delta\mathbf{M}, \quad \mathbf{C} = \mathbf{P}_0 + \delta\mathbf{C}, \tag{3}$$

where $\delta\mathbf{M} = \delta\mathbf{M}^T$ and $\delta\mathbf{C} \neq \delta\mathbf{C}^T$ are the small perturbations of matrices of stable conservative system $\mathbf{M}_0\ddot{\mathbf{q}} + \mathbf{P}_0\mathbf{q} = 0$ with the positive definite matrix $\mathbf{P}_0 = \mathbf{P}_0^T > 0$. If the eigenvalues $\lambda = i\omega$ of the conservative system are simple, the small perturbations of the matrices $\delta\mathbf{M}$ and $\delta\mathbf{C}$ result in a shift of the eigenvalues λ along the imaginary axis, and the system remains stable. Therefore, from the viewpoint of the stability, the case of multiple frequency ω is of interest.

We consider the most important case when the conservative system has the double frequency $\omega_0 > 0$. The eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ corresponding to it are determined from the equations and the normalization conditions

$$\mathbf{P}_0\mathbf{u}_i = \mu_0\mathbf{M}_0\mathbf{u}_i, \quad \mathbf{u}_i\mathbf{M}_0\mathbf{u}_j = \delta_{ij}, \quad \mu_0 = \omega_0^2, \quad i, j = 1, 2. \tag{4}$$

For perturbation (3), the eigenvalues μ and the corresponding eigenvectors $\mathbf{u}$ are found in the form of

$$\mu = \mu_0 + \delta\mu + \dots, \quad \mathbf{u} = \sum_{i=1}^{2}\alpha_i\mathbf{u}_i + \delta\mathbf{u} + \dots, \tag{5}$$

where two values of the correction $\delta\mu$ and the corresponding coefficients α_1 and α_2 are determined from the equations (see [5])

$$\sum_{j=1}^{2} a_{ij}\alpha_j = \delta\mu\alpha_i, \quad a_{ij} = \mathbf{u}_i^T\delta\mathbf{C}\mathbf{u}_j - \omega_0^2\mathbf{u}_i^T\delta\mathbf{M}\mathbf{u}_j, \quad i, j = 1, 2. \tag{6}$$

The system is stable if the correction $\delta\mu$ takes two various real values and is unstable when $\delta\mu$ are complex.

Hence, in the first approximation, we find the following condition of instability:

$$\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21} < 0. \quad (7)$$

This inequality determines the inner part of the cone in the space of coefficients a_{12}, a_{21}, and $(a_{11} - a_{22})/2$ (see Fig. 1a). The occurrence of the conical singularity in the three-dimensional space of parameters is natural because the double real semisimple eigenvalue μ in Eq. (2) defines the singularity of the codimension 3 in the bifurcation diagram of the family of nonsymmetric matrices [6].

The case of the frequency ω_0 of an arbitrary multiplicity r is considered similarly. In this case, the indexes i and j in Eqs. (4)–(6) vary from 1 to r. The case when $\delta\mathbf{C} \equiv \delta\mathbf{N} = -\delta\mathbf{N}^T$ is of interest; i.e., the pure circulatory forces are added to the conservative system with a multiple frequency. Then, the coefficients $a_{ij} = \mathbf{u}_i^T\delta\mathbf{N}\mathbf{u}_j$ have the property $a_{ij} = -a_{ji}$. Because in this case $i[a_{jk}]$ is the Hermitian matrix, all eigenvalues of the matrix $[a_{ij}]$ are simple or semisimple and located at the imaginary axis. Even if one of them is nonzero, the system becomes unstable. Thus, the conservative system with multiple frequency ω_0 is destabilized by small circulatory forces $\delta\mathbf{N}\mathbf{q}$ if $\mathbf{u}_i^T\delta\mathbf{N}\mathbf{u}_j \neq 0$ for certain eigenvectors corresponding to ω_0. It is easy to verify for particular examples that if $\mathbf{u}_i^T\delta\mathbf{N}\mathbf{u}_j = 0$ for all eigenvectors corresponding to ω_0, the system can remain stable. The destabilization of the potential system by the circulatory forces in the case when the multiplicity of ω_0 coincides with the system dimension is proved in [7].

EFFECT OF DISSIPATIVE FORCES

We investigate the effect of small dissipative forces described by the term $\delta\mathbf{D}\dot{\mathbf{q}}$ with the positive definite symmetric matrix $\delta\mathbf{D}$ added to the left-hand side of vibration system (1). Then (2) eigenvalue problem takes the form of

$$(\lambda^2\mathbf{M} + \lambda\delta\mathbf{D} + \mathbf{C})\mathbf{u} = 0. \quad (8)$$

It is well-known that the addition of small dissipative forces to a stable conservative system results in a change in $\mathrm{Re}\delta\lambda = -\frac{1}{2}\mathbf{u}^T\delta\mathbf{D}\mathbf{u} < 0$ for each simple eigenvalue $\lambda = i\omega$ with the eigenvector $\mathbf{u}$ (see, for example, [5]). As a result, the system becomes asymptotically stable. The same statement is true also in the presence of small nonconservative forces, when all eigenvalues λ are pure imaginary and simple (due to the smallness of the nonconservative forces, perturbation $\mathrm{Re}\delta\lambda$ is determined in the first approximation by the same formula). Thus, also in the presence of damping, the case of multiple frequency represents the major interest for the analysis of stability.

In the case of double frequency ω_0 of the conservative system, the analysis of stability is carried out similarly. In this case, in Eq. (6) written in terms of $\delta\mu = \delta(-\lambda^2) = -2i\omega_0\delta\lambda$, there is the dissipative term

$$\sum_{j=1}^{2}(2i\omega_0\delta\lambda\delta_{kj} + i\omega_0 d_{kj} + a_{kj})\alpha_j = 0, \quad k = 1, 2, \quad (9)$$

with the damping coefficients $d_{kj} = \mathbf{u}_i^T\delta\mathbf{D}\mathbf{u}_j$. The condition of asymptotic stability requires that $\mathrm{Re}\delta\lambda < 0$ for both roots $\delta\lambda$ of system (9). We write the characteristic equation of system (9) in the form of $\delta\lambda^2 + (D + iG)\delta\lambda + P + iN = 0$ and apply the Bilhartz criterion of stability for polynomials with the complex coefficients $D > 0$, $N^2 - GDN < D^2P$ (see, for example, [5]). The first condition $D = (d_{11} + d_{22})/2 > 0$ is fulfilled for the positive defined damping matrix $\delta\mathbf{D}$. The second inequality taken with the opposite sign gives the condition of instability in the form

$$\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21} + \omega_0^2 d^2(1-\xi_1^2-\xi_2^2) < \left(\xi_1\frac{a_{11}-a_{22}}{2} + \xi_2\frac{a_{12}+a_{21}}{2}\right)^2, \quad (10)$$

$$d = \frac{d_{11}+d_{22}}{2}, \quad \xi_1 = \frac{d_{11}-d_{22}}{d_{11}+d_{22}}, \quad \xi_2 = \frac{2d_{12}}{d_{11}+d_{22}}. \quad (11)$$

From the inequality $d_{11}d_{22} - d_{12}^2 > 0$ valid for the positive defined matrix $\delta\mathbf{D}$, the condition $\xi_1^2 + \xi_2^2 < 1$ follows. Thus, the last term in the left-hand side of Eq. (10) is positive. Instability condition (10) depends on three independent quantities d_{11}, d_{22}, $d_{12} = d_{21}$, a_{12}, a_{21}, and $a_{11} - a_{22}$. It agrees with the results of the singularity theory according to which the double semisimple complex eigenvalue $\lambda = i\omega_0$ defines the singularity of the codimension six for the families of real matrices [6].

At fixed d_{ij}, inequality (10) defines the hyperboloid in space $(a_{11} - a_{22}, a_{12}, a_{21})$, whose position substantially depends on the ratio between the damping coefficients ξ_1 and ξ_2 (Fig. 1b). In the case of $\xi_1 = \xi_2 = 0$, instability condition (10) becomes

$$\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}a_{21} + \omega_0^2 d^2 < 0,$$
$$d = d_{11} = d_{22}, \quad d_{12} = 0. \tag{12}$$

For $d \to 0$, this condition coincides with instability condition (7) for the system without damping. The presence of the damping $d > 0$ stabilizes the system (decreases the instability region). This case is characteristic for the systems where the multiple frequency ω_0 arises as a result of symmetry.

We now consider the case when $\xi_1 \neq 0$ or $\xi_2 \neq 0$. When the damping coefficients d_{ij} tend to zero with the fixed ratios between ξ_1 and ξ_2, the limiting instability region is obtained from Eq. (10) for $d = 0$.

This region is wider than the instability region of system (7) without damping because the right-hand side in Eq. (10) is positive. From here, it follows that taking into account small damping results in a finite (discontinuous) increase in the instability region dependent on the values of ξ_1 and ξ_2 (Fig. 1). In this case, the corresponding boundaries (cones) touch the rays lying in the plane $\xi_1(a_{11} - a_{22}) + \xi_2(a_{12} + a_{21}) = 0$ in which the term in the right-hand side of Eq. (10) is zero. This destabilization phenomenon is similar to the steplike increase in the combination resonance zone with adding a small damping (see [5, 8]). There is also an analogy to the Ziegler destabilization paradox, when the critical stability parameter for the circulatory system decreases by a finite value with taking into account an arbitrarily small damping (see [9, 4]).

It should be noted that V.V. Bolotin [4] investigated the stability of the system with two degrees of freedom and identical frequencies when adding small circulatory and dissipative forces. For comparison with the Bolotin results, we put that $a_{ij} = \beta b_{ij}\omega_0^2$, $a_{11} = a_{22} = 0$, $d_{11} = g_1$, $d_{22} = g_2$, and $d_{12} = 0$ with the loading parameter β and obtain the instability condition $\beta^2\omega_0^2 b_{12}b_{21} + g_1g_2 < 0$. This inequality agrees with formula (1.100) in [4] for the critical loading parameter

$$\beta_{**} = \frac{1}{\omega_0}\sqrt{-\frac{g_1g_2}{b_{12}b_{21}}} \tag{13}$$

except for the frequency ω_0 omitted in the denominator.

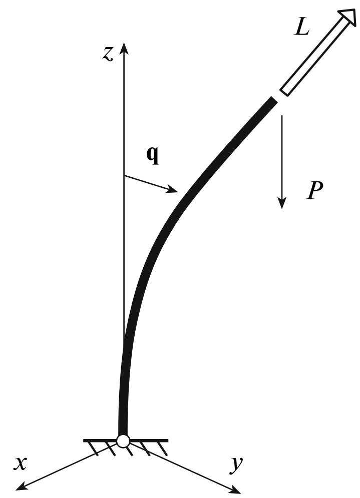

Fig. 2. Rod loaded with the axial force P and the tangential moment L.

NICOLAI'S PARADOX

We consider a straight elastic rod of length l of the linear mass m per unit length, with the constant transverse cross section having the principal moments of inertia J_x, J_y, and the elasticity modulus E. The rod is clamped at one end and loaded with the moment L and the axial force P at the free end. Following [4], we consider two types of the moment: the tangential moment directed along the tangent to the rod axis at the end (Fig. 2) and the axial moment parallel to the axis z. In the straight equilibrium position, the rod torsion is determined by the angle $\theta(z) = Lz/GI_d$, where GI_d is the torsion rigidity. Small vibrations of the rod about this equilibrium position are described by the equation

$$(E\mathbf{J}\mathbf{q}'')'' + L\mathbf{N}\mathbf{q}''' + P\mathbf{q}'' + m\ddot{\mathbf{q}} = 0 \tag{14}$$

for the deflection function $\mathbf{q}(z) = (q_x, q_y)^T$ determining the deviation of the rod axis in the direction of axes x and y. In Eq. (14), the dots and primes designate the derivatives with respect to time and z, respectively. The elastic forces are described by the first term $(E\mathbf{J}\mathbf{q}'')''$. Here the matrix of moments of inertia $\mathbf{J}(z)$ corresponds to the twisted equilibrium position, i.e., that calculated for the cross section of the rod turned through the angle $\theta(z)$:

$$\mathbf{J} = \mathbf{R}\begin{pmatrix} J_y & 0 \\ 0 & J_x \end{pmatrix}\mathbf{R}^T, \quad \mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{15}$$

The second term $L\mathbf{N}\mathbf{q}'''$ with the matrix $\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defines the generalized force corresponding to the moment L. In such a form, it was obtained in [4] for an axisymmetric rod. However, the form of this term is independent of the rod cross-section shape because the generalized force depends on the work of the moment applied to the rod end. The remaining two terms $P\mathbf{q}''$ and $m\ddot{\mathbf{q}}$ correspond to the axial force P and the force of inertia, respectively. The boundary conditions have the form [3, 4]

$$z = 0: \mathbf{q} = 0, \quad \mathbf{q}' = 0;$$
$$z = 1: \mathbf{q}'' = 0, \quad (E\mathbf{J}\mathbf{q}'')' + P\mathbf{q}' = 0$$
$$\text{(the tangential moment);} \qquad (16)$$
$$z = 1: E\mathbf{J}\mathbf{q}'' + L\mathbf{N}\mathbf{q}' = 0, \quad (E\mathbf{J}\mathbf{q}'')' + L\mathbf{N}\mathbf{q}'' + P\mathbf{q}' = 0$$
$$\text{(the axial moment).}$$

The solutions of Eqs. (14) and (16) in the form of $\mathbf{q}(s, t) = \mathbf{u}(s)\exp(\lambda t)$ are determined from the boundary-value problem

$$(E\mathbf{J}\mathbf{u}'')'' + L\mathbf{N}\mathbf{u}''' + P\mathbf{u}'' = m\mu\mathbf{u}, \quad \mu = -\lambda^2 \qquad (17)$$

with the same boundary conditions (16) for $\mathbf{u}(s)$. It is easy to show by the direct calculation that problems (17), (16), corresponding to the tangential and axial moments, are adjoint. Because the characteristic equations for the adjoint problems coincide, the stability problems for the cases of the tangential and axial moments are equivalent. Further, we consider the case of the tangential moment.

We write Eq. (14) with the boundary conditions in the dimensionless form

$$\ddot{\mathbf{q}} + (\mathbf{J}\mathbf{q}'')'' + L\mathbf{N}\mathbf{q}''' + P\mathbf{q}'' = 0, \qquad (18)$$

$$z = 0: \mathbf{q} = \mathbf{q}' = 0; \; z = 1: \mathbf{q}'' = (\mathbf{J}\mathbf{q}'')' + P\mathbf{q}' = 0, \qquad (19)$$

where we introduced the dimensionless variables and parameters

$$\tilde{z} = \frac{z}{l}, \quad \tilde{t} = t\sqrt{\frac{E(J_x + J_y)}{2m_0 l^4}}, \quad \tilde{\mathbf{q}} = \frac{\mathbf{q}}{l}, \quad \tilde{P} = \frac{2Pl^2}{E(J_x + J_y)},$$

$$\tilde{L} = \frac{2Ll}{E(J_x + J_y)}, \quad \chi = \frac{E(J_x + J_y)}{2GI_d}, \quad \sigma = \frac{J_y - J_x}{J_x + J_y}, \qquad (20)$$

$$\tilde{\mathbf{J}} = \mathbf{I} + \sigma\mathbf{K}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{K}(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

with the angle $\theta(\tilde{z}) = \chi\tilde{L}\tilde{z}$ and, then, omitted the tildes. We note that the quantity $\chi \sim 1$. Equation (17) is written in the dimensionless form

$$(\mathbf{J}\mathbf{u}'')'' + L\mathbf{N}\mathbf{u}''' + P\mathbf{u}'' = \mu\mathbf{u}. \qquad (21)$$

We investigate the system stability in the assumption of the smallness of the moment $|L| \ll 1$ and the asymmetry of the rod cross section $|\sigma| \ll 1$.

For the rod with identical principal moments of inertia of the cross section $\sigma = 0$ and in the absence of the tangential moment $L = 0$, all eigenvalues $\lambda = i\omega$ are double and semisimple due to the symmetry. The corresponding eigenforms are defined as

$$\mathbf{u}_1 = \begin{pmatrix} w(z) \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ w(z) \end{pmatrix}, \quad \int_0^1 w^2 dz = 1, \qquad (22)$$

where the last relation is the normalization condition. The function $w(z)$ is defined by the boundary-value problem

$$w'''' + Pw'' = \omega^2 w, \quad w(0) = w'(0) = 0, \quad w''(1) = w'''(1) + Pw'(1) = 0. \qquad (23)$$

The solutions of problem (23) with taking into account the first three boundary conditions have the form of

$$w = c(r_1^2\cosh r_1 + r_2^2\cos r_2)(r_1\sin r_2 z - r_2\sinh r_1 z) + cr_1r_2(r_1\sinh r_1 + r_2\sin r_2)(\cosh r_1 z - \cos r_2 z),$$
$$r_k = \sqrt{\frac{(-1)^k P + \sqrt{P^2 + 4\omega^2}}{2}}, \quad k = 1, 2, \qquad (24)$$

where the constant c is determined from normalization condition (22). The frequencies ω are found from the characteristic equation determined by the substitution of Eq. (24) into the fourth boundary condition in Eqs. (23):

$$2\omega^2 + (P^2 + 2\omega^2)\cosh r_1\cos r_2 - P\omega\sinh r_1\sin r_2 = 0. \qquad (25)$$

The critical Euler force $P_{cr} = \pi^2/4$ corresponds to the frequency $\omega = 0$; it is assumed that $P < P_{cr}$.

For small values of $|L| \ll 1$ and $|\sigma| \ll 1$, Eq. (21) contains the terms $\sigma\mathbf{K}(0)\mathbf{q}''''$ and $L\mathbf{N}\mathbf{q}'''$ of the first order of smallness (neglecting the terms of the second order of smallness in the matrix of moments of inertia: $\theta = \chi Lz \to 0$). The variation of the eigenvalue $\delta\mu$ is determined by Eq. (6), where the coefficients a_{ij} are calculated similarly by the substitution of Eq. (5) into Eqs. (21), (16), the multiplication of both sides of the differential equation on the left by $\mathbf{u}_i^T$ and the integration over z. As a result of several integrations by parts

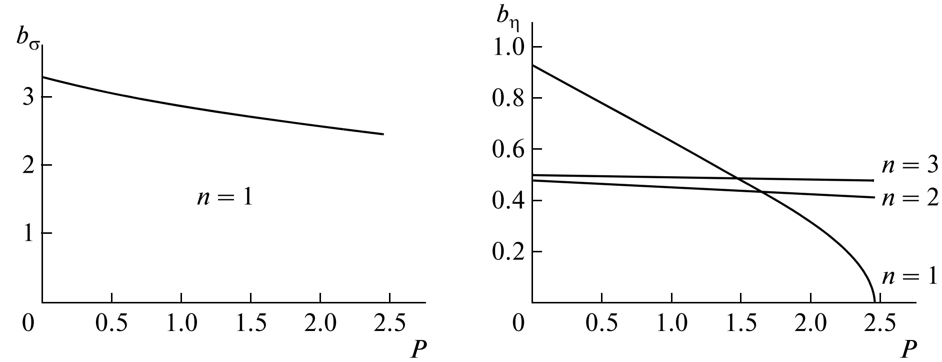

Fig. 3. Dependence of the coefficients b_σ and b_η in instability condition (31) on the dimensionless axial force P for various vibration modes n.

with using Eqs. (22), (23), the terms with $\delta\mathbf{u}$ vanish, and we find

$$[a_{ij}] = \frac{w'^2(1)}{2}\begin{pmatrix} b_\sigma\sigma & -L \\ L & -b_\sigma\sigma \end{pmatrix},$$
$$b_\sigma = \frac{2}{w'^2(1)}\int_0^1 w''^2 dz. \tag{26}$$

Then instability condition (7) for the vibration mode under consideration is written as

$$L^2 > b_\sigma^2\sigma^2. \tag{27}$$

The coefficient b_σ describes the effect of the asymmetry ($\sigma \neq 0$) leading to the positive critical moment value $L_{cr} = |b_\sigma\sigma| > 0$. From formula (27), it follows that the rod with identical moments of inertia ($\sigma = 0$) loses stability for an arbitrarily small value of the tangential moment L. This effect is known as the Nicolai paradox [1, 4]. The cross-section asymmetry gives the stabilizing effect for an arbitrary axial force $P < P_{cr}$.

As an example, we consider the rod with the elliptic cross section close to the circle of radius R, the semiaxes $R + \delta R$ and R. In this case, $J_y = \frac{\pi(R+\delta R)^3 R}{4}$, and $J_x = \frac{\pi(R+\delta R)R^3}{4}$, and the parameter $\sigma = \delta R/R$ in the first approximation in δR. For this rod in the absence of the longitudinal force $P = 0$, the formula for instability region (27) gives

$$L^2 > b_c^2\left(\frac{\delta R}{R}\right)^2 \tag{28}$$

with the coefficient $b_c = 3.26$. For the rod of the same cross section within the framework of the discrete model with two degrees of freedom, according to Nicolai [2], formula (28) with a smaller coefficient $b_d = 8/3$ is obtained. Thus, the use of the discrete model narrows the stability region.

We investigate the effect of the small dissipative forces described by additional terms in Eq. (18) in the form of $\eta_e\dot{\mathbf{q}}$ (external friction) and $\eta_i\dot{\mathbf{q}}''''$ (the Kelvin–Voigt model of internal friction) with dimensionless positive coefficients $\eta_e \ll 1$ and $\eta_i \ll 1$ on stability. With taking into account the smallness of σ in the term describing the internal friction, we put that $\mathbf{J} = \mathbf{I}$. Boundary condition (19) at the free end takes the form

$$z = 1: \mathbf{q}'' = 0, \quad (\mathbf{J}\mathbf{q}'')' + \eta_i\dot{\mathbf{q}}''' + P\mathbf{q}' = 0. \tag{29}$$

The coefficients of dissipative terms in Eq. (9) are found in the form of

$$d_{ij} = \eta\delta_{ij}, \quad \eta = \eta_e + \eta_i\int_0^1 w''^2 dz. \tag{30}$$

As the first approximation, the instability region is defined by inequality (12) written as

$$L^2 > b_\sigma^2\sigma^2 + b_\eta^2\eta^2, \quad b_\eta = \frac{2\omega}{w'^2(1)}. \tag{31}$$

The damping stabilizes the system because here we deal with the special case of $\xi_1 = \xi_2 = 0$ according to Eq. (11). The stabilizing effect of damping and of cross-section asymmetry was found for the first time by Nicolai [1–3] for the model of a massless elastic rod with the point mass at the end with two degrees of freedom (see also [4]). Relation (31) generalizes these effects to the case of distributed mass, the presence of axial force, and external and internal damping. The same formula defines the instability region for the rod loaded with the axial moment because the corresponding stability problems are equivalent.

The rod instability region is the sum of instability regions (31) calculated for all frequencies $0 < \omega_1 < \omega_2 < \ldots < \omega_n < \ldots$ determined by Eq. (25). The numerical calculations show that the coefficient b_σ grows quickly for all $P < P_{cr}$ with increasing the number n of the vibration mode. Therefore, the effect of the cross-section asymmetry on the stability in the absence of damping depends on the first vibration mode with the frequency ω_1. The coefficient b_σ for the first mode decreases with increasing the axial force P. The dependence of b_σ on P is shown in Fig. 3 on the left. The coefficient b_η depends on P and the numbers n of the vibration mode, the dependence on n being non-monotonic, and the limit $b_\eta \to 1/2$ for $n \to \infty$ (Fig. 3 on the right) takes place. For example, for $P = \sigma = 0$ and the presence of only external damping ($\eta_i = 0$), we obtain the instability condition $|L| > b_\eta \eta_e$, where $b_\eta = 0.9278$ for the first vibration mode, and $b_\eta = 0.482$ for the second one, and $b_\eta = 0.5$ for higher modes. In [4] from the analysis of the first mode, the stability condition for the rod loaded with the axial moment was obtained in the form of $|L| < 0.93\eta_e$ (in our notations). Due to equivalence of the stability problems, this inequality is valid also in the case of the tangential twisting moment. However, the correct stability condition is defined by the second mode and has the form of $|L| < 0.482\eta_e$.

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