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Paradox of Nicolai and related effects

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Abstract. The paper presents a general approach to the paradox of Nicolai and related effects analyzed as a singularity of the stability boundary. We study potential systems with arbitrary degrees of freedom and two coincident eigenfrequencies disturbed by small non-conservative positional and damping forces. The instability region is obtained in the form of a cone having a finite discontinuous increase in the general case when arbitrarily small damping is introduced. This is a new destabilization phenomenon, which is similar to well-known Ziegler's paradox or the effect of the discontinuous increase of the combination resonance region due to addition of infinitesimal damping. It is shown that only for specific ratios of damping coefficients, the system is stabilized due to presence of small damping. Then, we consider the paradox of Nicolai: the instability of a uniform axisymmetric elastic column loaded by axial force and a tangential torque of arbitrarily small magnitude. We extend the results of Nicolai showing that the column is stabilized by general small geometric imperfections and internal and external damping forces. It is shown that the paradox of Nicolai is related to the conical singularity of the stability boundary which transforms to a hyperboloid with the addition of small dissipation. As a specific example of imperfections, we study the case when cross-section of the column is changed from a circular to elliptic form.

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1. Introduction

In 1928, Nicolai [14] formulated a problem of stability of an elastic column with equal basic moments of inertia loaded by a tangential torque and axial force. For the case of cantilever boundary conditions, he found that there is no static form of equilibrium of the column except the straight one. Then, he studied stability of the straight form of equilibrium using dynamic method and came to the conclusion that it is unstable for arbitrary small magnitude of the torque. This effect is called *the paradox of Nicolai*. For the stability study, he used a discrete model with a lumped mass attached to the free end of a massless cantilever column. In the same paper, Nicolai introduced a small viscous damping and found that it has a stabilizing effect. In his next paper, Nicolai [15] reconsidered the stability problem of a cantilever column loaded by a tangential torque and introduced geometrical imperfections related to non-equal basic moments of inertia. He used the same discrete model for the stability study and came to the conclusion that the geometrical imperfections are also stabilizing. That was the beginning of the era of non-conservative stability problems. Later these two important papers [14,15] were included in the volume of selected works of Nicolai [16]. An account of these papers is given in the book by Bolotin [3].

In 1950–1960s, Ziegler [22,23] and Bolotin [3] explained absence of static forms of the loss of stability in several contemporary problems by non-conservative nature of loading leading to dynamic forms of instability (flutter). In particular, Ziegler [22] classified different loadings of elastic columns and shafts dividing them to conservative and non-conservative cases.

Recently, it became clear that destabilization paradoxes due to dissipation are associated with generic singularities of the stability boundary. This is Ziegler's destabilization paradox [7,10], destabilization

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effect for combination resonance [12,20], destabilization of a Hamiltonian system [9], see also [8,19] and the early study of the Routh-Hurwitz conditions [4]. We continue this list by showing that the paradox of Nicolai is related to the cone singularity of the stability boundary.

In this paper, we develop a general approach to the paradox of Nicolai, similar problems and related effects and find geometrical interpretation of them from the point of view of singularity theory. The paper is organized as follows. In Sect. 2, using perturbation method, we study potential systems with arbitrary degrees of freedom and two coincident eigenfrequencies disturbed by small non-conservative positional forces. In terms of non-conservative matrix of disturbance and eigenvectors corresponding to the double eigenfrequency, we derive an inequality showing that the instability region is located inside a conical singularity. Then, we extend the results to the case of arbitrary number of coincident frequencies and show that the system generally gets unstable when small purely circulatory forces with anti-symmetric matrix are added.

In Sect. 3, along with small non-conservative positional forces, we study influence of small damping to the stability boundary. Here we derive the instability region in the form of a cone becoming a hyperboloid with two sheets with a finite discontinuous increase of the instability region in the general case when arbitrary small damping is introduced. It is shown that only for a specific ratio of reduced damping coefficients the system is stabilized due to presence of small damping. This phenomenon can be treated as a new destabilization effect which is similar to well-known Ziegler's destabilization paradox due to presence of infinitesimal damping, see e.g., [3,19,21,23]. However, unlike in Ziegler's paradox, here we have a potential system with two equal eigenfrequencies, and the destabilization phenomenon is related to addition of both small non-conservative positional and damping forces.

In Sect. 4, we consider the stability problem of an elastic cantilever column with equal principal moments of inertia of cross-section, which is loaded by an axial force and tangential torque (the paradox of Nicolai). We derive general formulas for the instability region for small nonuniform geometric imperfections as well as internal and external damping. The Conclusion summarizes the results of the paper.

2. Destabilization of a conservative system by small circulatory forces

A linear vibrational system of arbitrary dimension with non-conservative positional forces can be written in the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0, \quad \mathbf{C} = \mathbf{P} + \mathbf{N}, \tag{2.1}$$

where $\mathbf{q}$ is the vector of generalized coordinates, $\mathbf{M}$ is the real symmetric positive definite mass matrix, the real matrices $\mathbf{P} = \mathbf{P}^T$ and $\mathbf{N} = -\mathbf{N}^T$ describe, respectively, potential and nonconservative (also called circulatory) forces. Eigenvalues of this system determining solutions of the form $\mathbf{q} = \mathbf{u}\exp(\lambda t)$ are found from the eigenvalue problem

$$\mathbf{C}\mathbf{u} = \mu\mathbf{M}\mathbf{u}, \quad \mu = -\lambda^2. \tag{2.2}$$

Clearly, $\lambda, -\lambda, \overline{\lambda}$, and $-\overline{\lambda}$ are eigenvalues, reflecting the time-reversal symmetry of real system (2.1). Hence, system (2.1) is stable if all the eigenvalues are purely imaginary and simple or semi-simple (having as many eigenvectors as its algebraic multiplicity), see, e.g., [17,19]. Equivalently, all eigenvalues μ in (2.2) must be simple or semi-simple, real, and positive. Simple real eigenvalues μ can leave the real axis only when becoming multiple. Thus, the stability boundary is characterized by zero or multiple positive eigenvalues μ.

In this paper, we are interested in the case when non-conservative forces are small and the corresponding conservative system (with $\mathbf{N} = 0$) is stable, i.e., $\mathbf{P}$ is positive definite. The oscillation frequencies ω of the conservative system are found as $\lambda = i\omega$. If all the frequencies are distinct, the addition of small non-conservative forces cannot destabilize the system, since simple eigenvalues λ cannot leave the imaginary axis. So, for the stability study, we are interested in the situation when some frequency ω and, hence, $\mu = \omega^2$ is multiple.

Consider first the most important case of a frequency with multiplicity $m = 2$. In the generic case, a double frequency appears at isolated points in two-parameter space of a conservative system (as a singularity of codimension 2 [1]). On the other hand, a double real semi-simple eigenvalue μ of (2.2) with a general non-symmetric real matrix $\mathbf{C} = \mathbf{P} + \mathbf{N}$ determines a singularity of codimension 3 and, thus, requires three parameters [2]. A double semi-simple eigenvalue determines a cone singularity of a surface corresponding to multiple eigenvalues [2] and, hence, of the stability boundary.

We consider a general small perturbation

$$\mathbf{M} = \mathbf{M}_0 + \delta\mathbf{M}, \quad \mathbf{C} = \mathbf{P}_0 + \delta\mathbf{C} \tag{2.3}$$

of the conservative system with a double frequency $\omega_0 > 0$. The two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of the unperturbed conservative system

$$\mathbf{M}_0\ddot{\mathbf{q}} + \mathbf{P}_0\mathbf{q} = 0 \tag{2.4}$$

are determined by the equations and normalization conditions

$$\mathbf{P}_0\mathbf{u}_i = \mu_0\mathbf{M}_0\mathbf{u}_i, \qquad \mathbf{u}_i\mathbf{M}_0\mathbf{u}_j = \delta_{ij}, \quad \mu_0 = \omega_0^2, \quad i,j = 1,2. \tag{2.5}$$

The leading terms of perturbation of the double eigenvalue μ_0 and the corresponding eigenvector can be written as [19]

$$\mu = \mu_0 + \delta\mu + \cdots, \quad \mathbf{u} = \sum_{i=1}^{2}\alpha_i\mathbf{u}_i + \delta\mathbf{u}\cdots, \tag{2.6}$$

where α_1 and α_2 are unknown coefficients. We substitute (2.3) and (2.6) into (2.2) and multiply the equation by $\mathbf{u}_1^T$ or $\mathbf{u}_2^T$ on the left. Using the relation $\mathbf{u}_i^T\mathbf{P}_0 = \mu_0\mathbf{u}_i^T\mathbf{M}_0$ following from (2.5) for symmetric matrices $\mathbf{M}_0$ and $\mathbf{P}_0$, we get rid of the terms containing $\delta\mathbf{u}$. Then, for the first-order terms, we obtain equations

$$\sum_{j=1}^{2} a_{ij}\alpha_j = \delta\mu\alpha_i, \quad a_{ij} = \mathbf{u}_i^T\delta\mathbf{C}\mathbf{u}_j - \omega_0^2\mathbf{u}_i^T\delta\mathbf{M}\mathbf{u}_j, \quad i,j = 1,2. \tag{2.7}$$

We see that two values of $\delta\mu$ describing the bifurcation of the double eigenvalue μ_0 are found as eigenvalues of the matrix $[a_{ij}]$ as

$$(\delta\mu)^2 - (a_{11} + a_{22})\delta\mu + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{2.8}$$

The system is stable when $\delta\mu$ takes two distinct real values. The system gets unstable when $\delta\mu$ is complex, i.e., the discriminant of the characteristic equation (2.8) is negative

$$\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}a_{21} < 0. \tag{2.9}$$

This inequality determines the internal part of a cone in the space $(a_{12}, a_{21}, (a_{11} - a_{22})/2)$, see Fig. 1a. Note that the conical singularity of the stability boundary associated with a double semi-simple eigenvalue was studied in [2,6,11,18,19], see also [5].

The case of frequency ω_0 of arbitrary multiplicity r is studied similarly. We just have to let the indices i, j in (2.5)–(2.7) vary from 1 to r. It is interesting to look at a specific case when small purely circulatory forces $\delta\mathbf{N}\mathbf{q}$ with $\delta\mathbf{N} = -\delta\mathbf{N}^T$ are added to conservative system (2.4). In this case, the coefficients $a_{ij} = \mathbf{u}_i^T\delta\mathbf{N}\mathbf{u}_j$ satisfy the anti-symmetric relation $a_{ij} = -a_{ji}$ and, thus, $[a_{ij}]$ multiplied by the imaginary unit i is the Hermitian matrix. Hence, all eigenvalues $\delta\mu$ of the matrix $[a_{ij}]$ are simple or semi-simple and lie on the imaginary axis. Unless all of them are zero, the system gets unstable. We conclude that

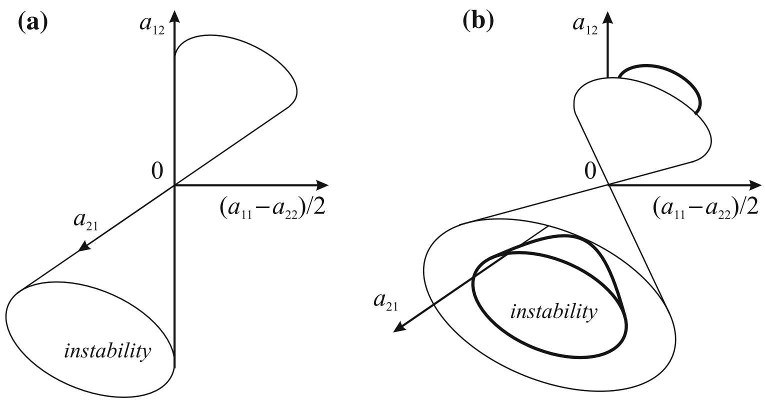

FIG. 1. **a** The cone singularity of the instability domain in the case of a perturbed conservative system with a double frequency. The coordinates are related to matrix perturbations by Eq. (2.7). **b** The stability boundary for a system with finite damping (*bold lines*) and infinitesimal damping (*thin lines*)

addition of small circulatory forces $\delta\mathbf{Nq}$ to a conservative system with a multiple frequency ω_0 destabilizes the system if $\mathbf{u}_i^T\delta\mathbf{N}\mathbf{u}_j \neq 0$ for some eigenvectors corresponding to ω_0. When $\mathbf{u}_i^T\delta\mathbf{N}\mathbf{u}_j = 0$ for all the eigenvectors corresponding to ω_0, i.e., the circulatory forces do not couple the modes corresponding to ω_0, the system may remain stable as one can check on specific examples. Note that the case when the multiplicity r is equal to the number of degrees of freedom has been analyzed in [13].

3. Influence of small damping

Consider the case when small damping forces $\delta\mathbf{D}\dot{\mathbf{q}}$ with positive definite symmetric matrix $\delta\mathbf{D}$ are added to the left-hand side of (2.1). The eigenvalue problem (2.2) becomes

$$(\lambda^2\mathbf{M} + \lambda\delta\mathbf{D} + \mathbf{C})\mathbf{u} = 0. \tag{3.1}$$

We assume that both dissipative forces and variations (2.3) are small, though not requiring any relation between them. It is well-known that adding small dissipative forces to a stable conservative system gives the negative real perturbation $\delta\lambda = -\mathbf{u}^T\delta\mathbf{D}\mathbf{u}/2$ to each simple eigenvalue $\lambda = i\omega$ with the eigenvector $\mathbf{u}$, and the system becomes asymptotically stable, see, e.g., [19]. The same statement is valid when small nonconservative forces (2.3) are present and all the eigenvalues λ are simple. Indeed, in this case, the eigenvalues λ of the system without damping lie on the imaginary axis, while their variation due to dissipation $\delta\lambda \approx -\mathbf{u}^T\delta\mathbf{D}\mathbf{u}/2 < 0$ is weakly affected by small nonconservative forces, making the system asymptotically stable. Thus, in the presence of damping, the interesting cases are related again to multiple frequencies.

When the initial potential system (2.4) has a double frequency ω_0, the stability analysis is carried out by adding a dissipative term to (2.7), written for $\delta\mu = \delta(-\lambda^2) = -2i\omega_0\,\delta\lambda$ as

$$\sum_{j=1}^{2}(2i\omega_0\,\delta\lambda\,\delta_{kj} + i\omega_0 d_{kj} + a_{kj})\,\alpha_j = 0, \quad k = 1, 2 \tag{3.2}$$

with the damping coefficients $d_{kj} = \mathbf{u}_k^T\delta\mathbf{D}\mathbf{u}_j$. The asymptotic stability condition is determined by requiring $Re\,\delta\lambda < 0$ for both eigenvalues $\delta\lambda$ of system (3.2). Writing the characteristic equation of (3.2) in the form $\delta\lambda^2 + (D + iG)\delta\lambda + P + iN = 0$, we can use the Bilharz stability conditions $D > 0, N^2 - GDN < D^2P$ (see, e.g., [19]). The first condition $D = (d_{11} + d_{22})/2 > 0$ is valid for the positive definite damping matrix $\delta\mathbf{D}$. Modifying the second condition and changing the inequality sign, we obtain the instability condition as $P + G^2/4 < (N/D - G/2)^2$. This instability condition up to a positive factor is written in terms of the original system (3.2) as

$$\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21} + \omega_0^2 d^2(1-\eta_1^2-\eta_2^2) < \left(\eta_1\frac{a_{11}-a_{22}}{2} + \eta_2\frac{a_{12}+a_{21}}{2}\right)^2, \tag{3.3}$$

$$d = \frac{d_{11}+d_{22}}{2}, \quad \eta_1 = \frac{d_{11}-d_{22}}{d_{11}+d_{22}}, \quad \eta_2 = \frac{2d_{12}}{d_{11}+d_{22}}. \tag{3.4}$$

The ratios defined in (3.4) satisfy the inequality $\eta_1^2+\eta_2^2 < 1$, which is equivalent to $d_{11}d_{22} - d_{12}^2 > 0$ and follows from positive definiteness of the damping matrix $\delta\mathbf{D}$. This means that the last term in the left-hand side of inequality (3.3) is positive. The instability condition (3.3) depends on six independent quantities $d_{11}, d_{22}, d_{12} = d_{21}, a_{12}, a_{21}$, and $a_{11} - a_{22}$. This agrees with the fact that the double semi-simple eigenvalue $\lambda = i\omega_0$ determines a singularity of codimension 6 in general systems of linear differential equations [2].

For fixed d_{ij}, inequality (3.3) determines a hyperboloid of two sheets in the space $(a_{11} - a_{22}, a_{12}, a_{21})$ depending essentially on the ratios η_1 and η_2 of damping coefficients, Fig. 1b. When $\eta_1 = \eta_2 = 0$, condition (3.3) takes the form

$$\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21} + \omega_0^2 d^2 < 0 \quad (d = d_{11} = d_{22}, \quad d_{12} = 0). \tag{3.5}$$

When $d \to 0$, this inequality coincides with the instability condition (2.9) for a system with no damping. The damping $d > 0$ has stabilizing effect decreasing the instability region.

Otherwise, if $\eta_1 \neq 0$ or $\eta_2 \neq 0$, the limiting instability region with infinitely small damping is obtained by taking $d = 0$ in (3.3). It is larger than the instability region with zero damping given by (2.9), since the right-hand side in (3.3) remains finite and positive when the damping coefficients d_{ij} tend to zero with the fixed ratios η_1 and η_2. So, based on (3.3), we conclude that when arbitrarily small dissipation is introduced, the instability region undergoes a finite (discontinuous) increase depending on η_1 and η_2, Fig. 1b.

The described destabilization phenomenon is very similar to the discontinuous increase of a combination resonance region due to infinitesimal damping in the theory of parametric resonance, see, e.g., [19]. There is also analogy with Ziegler's destabilization paradox, when a critical stability parameter of a non-conservative system with positional forces drops down with introduction of infinitely small damping, see, e.g., [3,19,21,23]. However, unlike in Ziegler's paradox, here we have a potential system with two equal eigenfrequencies, and the destabilization phenomenon is related to introduction of both small non-conservative and damping forces. These two paradoxes are also different from the point of view of singularity theory: Nicolai's paradox is related to a double semi-simple eigenfrequency (the conical singularity) while Ziegler's paradox is associated with a double eigenfrequency with the Jordan block (the singularity known as Whitney's umbrella) [19].

Note that Bolotin [3] studied stability of a system with two degrees of freedom and equal eigenfrequencies under the action of small non-conservative positional and dissipative forces. We can compare the results if we put in inequality (3.3) $a_{ij} = \beta b_{ij}\omega_0^2, a_{11} = a_{22} = 0$ and $d_{11} = g_1, d_{22} = g_2, d_{12} = 0$ with β being the stability parameter. Then, we derive the instability region in the form

$$\beta^2\omega_0^2 b_{12}b_{21} + g_1 g_2 < 0, \tag{3.6}$$

which agrees with formula (1.100) in Bolotin's book [3] for the critical stability parameter of the damped system

$$\beta_{**} = \frac{1}{\omega_0}\sqrt{-\frac{g_1 g_2}{b_{12}b_{21}}} \tag{3.7}$$

except missing ω_0 in the denominator.

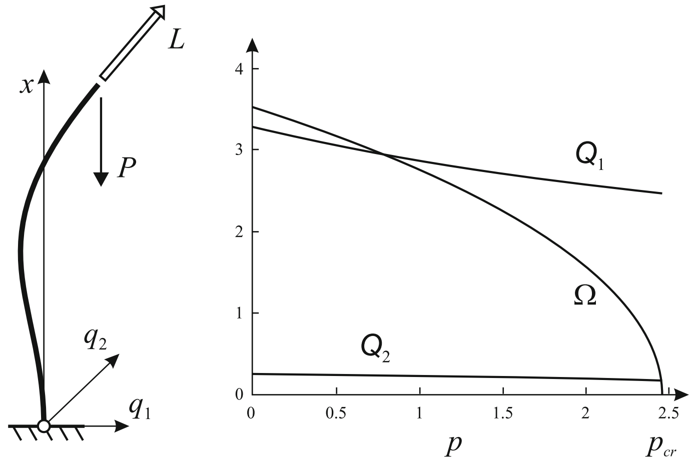

FIG. 2. A column loaded by an axial force P and tangential torque L. Dependence of the dimensionless eigenfrequency Ω and coefficients (4.20) in the instability condition (4.19) on the dimensionless axial force p

4. Stability of a column loaded by an axial force and tangential torque

Consider a straight cantilever elastic column of length l loaded at the free end by a tangential torque L and an axial force P, see Fig. 2. The column has a straight and twisted equilibrium. Small vibrations of the column near the equilibrium position are described by the equation in vector form [3]

$$m\ddot{\mathbf{q}} + E(\mathbf{J}\mathbf{q}'')'' + L\mathbf{N}\mathbf{q}''' + P\mathbf{q}'' = 0, \tag{4.1}$$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{12} & J_{22} \end{pmatrix}, \tag{4.2}$$

where the prime denotes the derivative with respect to the axial coordinate x, q_1 and q_2 are the column deflections in two lateral directions, E is Young's modulus, $m(x)$ is the mass per unit length, $\mathbf{J}(x)$ is the matrix of moments of inertia of cross-section for the twisted column. The boundary conditions for the cantilever are

$$x = 0: \ \mathbf{q} = \mathbf{q}' = 0; \quad x = l: \ \mathbf{q}'' = E(\mathbf{J}\mathbf{q}'')' + P\mathbf{q}' = 0. \tag{4.3}$$

Solutions $\mathbf{q}(x,t) = \mathbf{u}(x)\exp(\lambda t)$ are determined by the eigenvalue problem

$$E(\mathbf{J}\mathbf{u}'')'' + L\mathbf{N}\mathbf{u}''' + P\mathbf{u}'' = \mu m\mathbf{u}, \quad \mu = -\lambda^2, \tag{4.4}$$

with the same boundary conditions (4.3) for $\mathbf{u}(x)$.

Note that the boundary value problem adjoint to (4.4), (4.3) has the same form with the only difference in the boundary condition at the free end

$$x = l: \ E\mathbf{J}\mathbf{q}'' + L\mathbf{N}\mathbf{q}' = E(\mathbf{J}\mathbf{q}'')' + L\mathbf{N}\mathbf{q}'' + P\mathbf{q}' = 0. \tag{4.5}$$

This boundary condition corresponds to the *axial moment* L parallel to the axis x and applied at the end of the column [3]. Since the characteristic equations for adjoint problems are the same, the stability problems for columns loaded by tangential and axial moments are equivalent.

For the uniform column with equal principal moments of inertia of cross-section, we have $\mathbf{J} = J_0\mathbf{I}$ and $m = m_0$ with the 2×2 identity matrix $\mathbf{I}$ and constants m_0, J_0. When $L = 0$ and P does not exceed the critical Euler value, all the eigenvalues $\lambda = i\omega_0$ for this column are double semi-simple and determined by the eigenvalue problem

$$\begin{aligned} &EJ_0 w'''' + Pw'' = \omega_0^2 m_0 w, \\ &w(0) = w'(0) = 0, \quad w''(l) = EJ_0 w'''(l) + Pw'(l) = 0. \end{aligned} \tag{4.6}$$

The two eigenmodes corresponding to the frequency ω_0 are

$$\mathbf{u}_1 = \begin{pmatrix} w(x) \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ w(x) \end{pmatrix}, \quad \int_0^l w^2 \mathrm{d}x = \frac{1}{m_0}, \tag{4.7}$$

where the last expression is the normalization condition. Solution to system (4.6) satisfying the first three boundary conditions is

$$\begin{aligned} w = {} & c(r_1^2 \cos r_1 + r_2^2 \cosh r_2)(r_2 \sin r_1 \xi - r_1 \sinh r_2 \xi) \\ & - c r_1 r_2 (r_1 \sin r_1 + r_2 \sinh r_2)(\cos r_1 \xi - \cosh r_2 \xi), \quad \xi = x/l \end{aligned} \tag{4.8}$$

with the dimensionless coefficients

$$\begin{aligned} &r_1 = \sqrt{\frac{p + \sqrt{p^2 + 4\Omega^2}}{2}}, \quad r_2 = \sqrt{\frac{-p + \sqrt{p^2 + 4\Omega^2}}{2}}, \\ &p = \frac{Pl^2}{EJ_0}, \quad \Omega = \omega_0 \sqrt{\frac{m_0 l^4}{EJ_0}}. \end{aligned} \tag{4.9}$$

The dimensionless eigenfrequencies Ω are determined by the characteristic equation, which is found by substituting (4.8) into the last boundary condition in (4.6) as

$$2\Omega^2 + (p^2 + 2\Omega^2) \cos r_1 \cosh r_2 - p\Omega \sin r_1 \sinh r_2 = 0. \tag{4.10}$$

Finally, the normalization coefficient c is determined by the last equality in (4.6). Note that the critical Euler force is $P_{\mathrm{cr}} = \pi^2 EJ_0/(4l^2)$, which corresponds to $\Omega = 0, p_{\mathrm{cr}} = \pi^2/4, r_1 = \pi/2$ and $r_2 = 0$. In this section, we assumed that $P < P_{\mathrm{cr}}$.

Consider small tangential torque L and small geometric imperfections of the column leading to variations $\mathbf{J}(x) = J_0 \mathbf{I} + \delta \mathbf{J}(x)$ and $m(x) = m_0 + \delta m(x)$. Then, we obtain Eq. (2.7). The coefficients a_{ij} are computed similarly, substituting (2.6) into (4.3), (4.4), multiplying the differential equation by $\mathbf{u}_i^T$ on the left, and then integrating with respect to x. After several integrations by parts and using (4.6), (4.7), we get rid of the terms containing $\delta \mathbf{u}$. Thus, we obtain

$$[a_{ij}] = \int_0^l (E\delta \mathbf{J} w''^2 - \omega_0^2 \delta m w^2) \mathrm{d}x - L\mathbf{N} \int_0^l w' w'' \mathrm{d}x. \tag{4.11}$$

With this expression, the first-order instability condition (2.9) is written as

$$L^2 > b_1^2 + b_2^2, \tag{4.12}$$

where

$$\begin{aligned} &b_1 = \frac{1}{2\beta} \int_0^l E(\delta J_{11} - \delta J_{22}) w''^2 \mathrm{d}x, \quad b_2 = \frac{1}{\beta} \int_0^l E\delta J_{12} w''^2 \mathrm{d}x, \\ &\beta = \int_0^l w' w'' \mathrm{d}x = \frac{w'^2(l)}{2}. \end{aligned} \tag{4.13}$$

The quantities b_1 and b_2 describe the effect of geometric imperfections of the column. In the space (b_1, b_2, L), the instability region corresponds to the interior of the cone. Note that the instability region of the column is determined by the sum of instability regions corresponding to each eigenfrequency $\Omega > 0$ given by (4.10).

Formula (4.12) shows that the perfect column ($b_1 = b_2 = 0$) is destabilized by an arbitrarily small tangential torque L. This effect is known as the paradox of Nicolai [3,14]. However, the column is stabilized by general geometric imperfections at an arbitrary axial force P, less than the critical Euler value.

To determine the effect of dissipative forces, we add on the left side of equation of motion (4.1) the dissipative term $\eta E(\mathbf{J}\dot{\mathbf{q}}'')''$ with the coefficient $\eta > 0$ according to the Kelvin–Voigt model. The boundary conditions (4.3) at the free end $x = l$ change to

$$x = l: \quad \mathbf{q}'' = E(\mathbf{J}(\mathbf{q}'' + \eta\dot{\mathbf{q}}''))' + P\mathbf{q}' = 0. \tag{4.14}$$

Assuming that η is small, the coefficients of corresponding dissipative terms in (3.2) are found as

$$d_{ij} = d\delta_{ij}, \quad d = \eta E J_0 \int_0^l w''^2 dx. \tag{4.15}$$

We see from (3.4) that $\eta_1 = (d_{11} - d_{22})/(d_{11} + d_{22}) = 0$ and $\eta_2 = 2d_{12}/(d_{11} + d_{22}) = 0$. Hence, this is the special case when damping stabilizes the system. For small damping, the instability region takes the form of hyperboloid (3.5) written as

$$L^2 > b_1^2 + b_2^2 + (\omega_0 d/\beta)^2. \tag{4.16}$$

The same effect takes place for small external viscous damping given by the additional term $\gamma\dot{\mathbf{q}}$ in Eq. (4.1). This leads to the dissipation coefficients $d_{ij} = \delta_{ij}\gamma/m_0$ in (3.2). In this case, instead of the last term in (4.16), we get $(\omega_0\gamma)^2/(\beta m_0)^2$.

It is easy to extend formula (4.16) for the case when both external and internal damping are present. Using (3.5), we obtain the formula

$$L^2 > b_1^2 + b_2^2 + \omega_0^2(d + \gamma/m_0)^2/\beta^2 \tag{4.17}$$

showing the stabilizing effect of external and internal small damping. Note that the stabilizing effects of small external viscous damping and, separately, of geometrical imperfections were recognized by Nicolai [14–16] for a massless column with a lumped mass attached to the free end, see also [3]. Here with formula (4.17), we summarize and extend those two effects for the continuous model of the column with an axial force, general imperfections, and for presence of both external and internal dissipative forces. The same formula is valid for the case of axial torque instead of the tangential torque, since the corresponding eigenvalue problems are adjoint.

As an example, we consider the case when a column of circular cross-section with the radius R is slightly changed to elliptic cross-section with the semi-axes $R + \delta R$ and R. Then, the change of the cross-sectional moments of inertia $J_0 = \pi R^4/4$ in the first approximation is given by

$$\begin{aligned} \delta J_{11} &= \frac{\pi}{4}((R + \delta R)^3 R - R^4) \approx 3J_0\frac{\delta R}{R}, \\ \delta J_{22} &= \frac{\pi}{4}(R^3(R + \delta R) - R^4) \approx J_0\frac{\delta R}{R} \end{aligned} \tag{4.18}$$

with $J_{12} = \delta J_{12} = 0$. Using relations (4.13), (4.15), (4.18) and the last equality in (4.7), we write formula (4.17) for the instability region in the dimensionless form

$$\left(\frac{lL}{EJ_0}\right)^2 > Q_1^2\left(\frac{\delta R}{R}\right)^2 + \left(\Omega Q_1\frac{\eta}{l^2}\sqrt{\frac{EJ_0}{m_0}} + \Omega Q_2\frac{\gamma l^2}{\sqrt{m_0 EJ_0}}\right)^2. \tag{4.19}$$

The dimensionless frequency Ω is defined in (4.9) and

$$Q_1 = 2\left(\frac{dw}{d\xi}\right)^{-2}_{\xi=1} \int_0^1 \left(\frac{d^2w}{d\xi^2}\right)^2 d\xi, \quad Q_2 = 2\left(\frac{dw}{d\xi}\right)^{-2}_{\xi=1} \int_0^1 w^2 d\xi. \tag{4.20}$$

According to (4.8)–(4.10), the dimensionless frequencies Ω and coefficients Q_1 and Q_2 depend only on the dimensionless axial force p. Their graphs computed for the first mode (first positive root Ω) are presented in Fig. 2. In particular, at $p = 0$, we have $\Omega = 3.5160, Q_1 = 3.2622$ and $Q_2 = 0.2639$. For higher modes, the coefficients Q_1 are much larger. However, for increasing mode number n, the product ΩQ_2 increases or decreases depending on p and n, and tends to the constant value $1/2$ when $n \to \infty$. Thus, the instability region is determined by the first mode unless the external damping is dominant. In the latter case, the instability region may be determined by higher modes.

For no axial force, no internal damping and no geometrical imperfections, $p = \eta = \delta R = 0$, the minimum of the product $\Omega Q_2 = 0.482$ is attained at the second mode. Then, the instability region determined by (4.19) takes the form

$$|L| > 0.482\,\gamma l \sqrt{\frac{EJ_0}{m_0}}. \tag{4.21}$$

Note that the equivalent case of a column loaded by an axial moment and small external damping was studied by Bolotin [3, p. 138] and his stability condition in our notation is $|L| < 0.93\gamma l\sqrt{EJ_0/m_0}$. This condition was calculated for the first mode with $\Omega Q_2 = 0.9278$. However, the correct stability bound (4.21) is determined by the second mode.

5. Conclusion

In this paper, we developed a general approach to the paradox of Nicolai and related effects analyzed from the point of view of singularity theory. Geometrical interpretation of the obtained results is that the boundary of the instability region represents a conical surface in the reduced three-dimensional space of nonconservative disturbance parameters. It is shown that damping forces change the conical instability region to a hyperboloid with two sheets increasing or decreasing the instability region. We confirmed and extended the results of Nicolai showing that the uniform cantilever column with equal principal moments of inertia loaded by an axial force loses stability under the action of an arbitrary small tangential torque, but it is stabilized by small geometric imperfections and internal and external damping forces. The same result holds when the tangential torque is substituted by the axial torque, since the corresponding eigenvalue problems are adjoint.

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