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Combined effect of spatially fixed and rotating asymmetries on stability of a rotor

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A R T I C L E I N F O

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A B S T R A C T

In this paper we study the combined effects of bearing and rotor asymmetry on the stability of the classical Laval rotor using analytical techniques. This setting intrinsically features equations of motion with periodic coefficients. We obtain closed form approximations for the stability boundaries which give insights in the interaction of different effects which are elsewhere mostly considered in isolated form. Using a classical two degree of freedom model of a rotor we investigate dissipative and follower force type of forcing in combination with internal and external damping in the presence of both rotating and stationary asymmetries. Since in many technical applications the resulting forces are small compared to the elastic restoring forces of the corresponding symmetric system, the former are treated as perturbations. The results are benchmarked with numerical simulations indicating the range of validity of the approximations. Our model is designed to illustrate the most generic effects originating from the coupling of rotor and bearing asymmetries, which cannot be captured in models with constant coefficients.

1. Introduction

Rotordynamics is a fascinating field of mechanics with many interesting effects and phenomena. Typical effects are usually explained with minimal phenomenological models that are designed to be easily solvable [1–4], which means that usually a formulation with equations of motion with constant coefficients is desired [5–8]. The easiest model is probably the classical Laval rotor with only two degrees of freedom [9]. In order to keep the equations simple usually symmetry of the bedding or the rotor is assumed, leading to equations with constant coefficients in a specific coordinate frame. In this case the instability phenomena can be classified and studied in a general form [10]. However, it is intuitively clear that for an asymmetric rotor with asymmetric beddings no straightforward formulation of the equations of motion with constant coefficients is possible. The equations of motion for such a configuration of a Laval rotor can for example be found in [11]; numerical and experimental study of a rotor with flexible base was carried out in [12]. Periodic coefficients are also unavoidable if an asymmetric rotor interacts with stationary contacts as, for example, in brakes or clutches. For these systems intrinsically damping and follower type forces arise. The occurrence of periodic coefficients in the equations of motion combines the phenomena of parametric excitation and self-excitation. One of the first authors to discover this was

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Tondl [13] who showed that parametric excitation can be used on purpose to avoid self-excited vibrations [14,4]. Our paper therefore aims in the same direction, however, using different mathematical approaches.

It is well known that asymmetry of rotors can have destabilizing effects which occur for supercritical speed ranges. It has been shown that in other settings asymmetries in rotor systems have a stabilizing effect especially in the presence of nonconservative forcing [15–17]. Whereas for rotors with degrees of freedom parallel to the axis of rotation, as for example in rotating plates, a rotating asymmetry is helpful for stabilization, in case of a rotor with degrees of freedom orthogonal to the axis of rotation, asymmetries in the bearings stabilize the system [15–17].

These observations give the motivation to analyze the combined effect of stationary (spatially fixed) and rotating (body fixed) asymmetries in a rotor system. This is especially interesting for the Laval rotor, since the static asymmetry can stabilize and rotating asymmetry destabilize the system in context to inner damping. Therefore, in this paper we consider an asymmetric Laval rotor in an asymmetric bedding with internal and external damping as well as frictional contact forces. Since the damping and contact forces are usually small compared to elastic restoring forces, tendencies and crucial effects can be simultaneously studied using analytical perturbation theory for eigenvalues [18]. The methods are used in the context of Floquet theory for systems of equations which intrinsically feature time periodic coefficients. The results can be seen as a follow up of the analysis performed in [17], where the first observations were made on the combination of rotating and nonrotating asymmetries in rotors. Since the structure of the equations of motion depends on the definition of degrees of freedom (for example, for problem formulation in stationary or rotating coordinates) we take special care to obtain a coordinate invariant problem formulation in order to isolate physical effects from effects of representation.

We provide an analytical treatment for stability of a rotor with stationary and rotating asymmetries in the whole range of rotation speeds. In this theory, the resonant speeds represent the main mathematical novelty and difficulty. We show that due to intrinsic time-periodicity of system coefficients, the resonances are associated to semi-simple eigenvalues of multiplicity four. This represents an essentially higher degree of degeneracy as compared to the parametric resonance theory, where instability zones (tongues) result from double eigenvalues [18–20]. We show that the perturbation equations can be resolved due to their specific structure associated with the axially symmetric unperturbed equations. At the same time, the quadruple eigenvalue leads to a highly nonlinear form of first-order stability conditions in terms of problem parameters, providing a new type of trefoil-shaped resonance zones instead of classical instability tongues. We use the advantage of having a relatively simple system, which facilitates the comparison of analytical results with accurate numerical computation of stability regions. We expect that the proposed analytical stability theory can be generalized to systems of higher dimension, where detailed numerical multi-parameter analysis is a difficult task [21–24].

The paper is organized as follows. We describe the model in Section 2. Section 3 formulates the Floquet theory for the perturbation problem. Bifurcation of Floquet multipliers is studied in Section 4. Stability conditions are derived and analyzed for non-resonant and resonant rotation speeds in Sections 5 and 6, respectively. The results are summarized in Section 7.

2. Model

We consider a Laval rotor on a massless asymmetric elastic shaft (bending stiffnesses EI_1 and EI_2) that is bedded on the ground with asymmetric springs (stiffnesses k_1 and k_2) as shown in Fig. 1. The mass of the rotor is assumed to have only translatory degrees of freedom and a negligible unbalance. Note that the latter is not a strong limitation, because a small unbalance can be considered by studying the stability of a particular solution in a similar way.

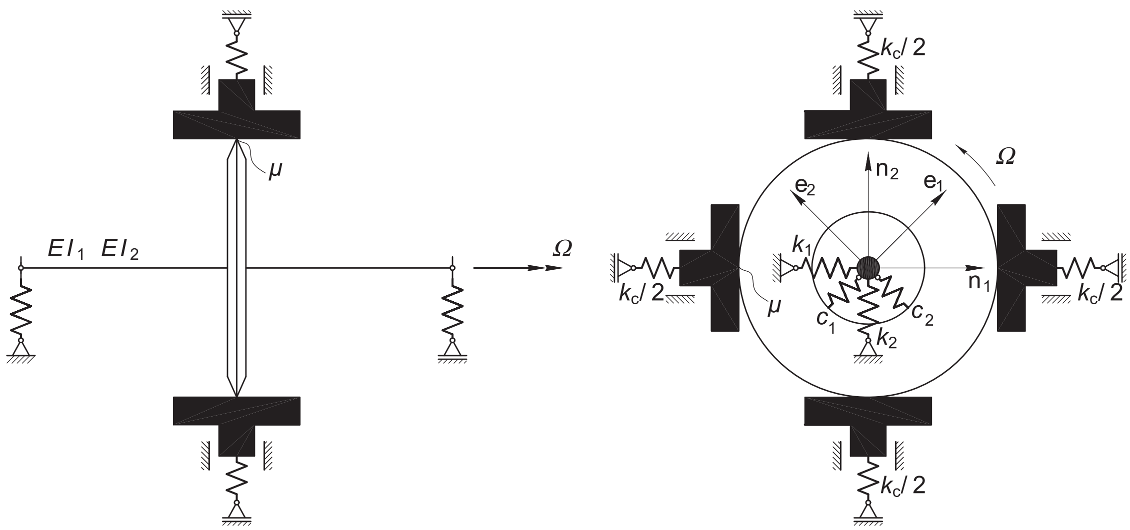

Fig. 1. Laval rotor with inner and outer stiffnesses and contact forces.

For the derivation of the equations of motion we use a stationary coordinate system spanned by the unit vectors $\vec{n}_1$, $\vec{n}_2$ and a coordinate system $\vec{e}_1$, $\vec{e}_2$ rotating with the rotor with constant angular velocity $\Omega = \dot{\Phi} > 0$. The equations of motion obtained from the principl

$$\left(\vec{\vec{M}} \cdot \ddot{\vec{r}} + \vec{\vec{K}} \cdot \vec{r}\right) \cdot \delta\vec{r} = (\vec{F}_d + \vec{F}_c) \cdot \delta\vec{r}, \tag{1}$$

where $\vec{r}$ is the position vector of the rotor. On the left we have the inertia tensor given by

$$\vec{\vec{M}} = m\vec{\vec{I}}, \quad \vec{\vec{I}} = \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2, \tag{2}$$

and the stiffness tensor $\vec{\vec{K}}$ originating from an elastic bedding.

In order to obtain $\vec{\vec{K}}$ we note that it is composed of the alignment of a stationary elastic bedding

$$\vec{\vec{K}}_{\text{stat}} = (k_1\vec{n}_1\vec{n}_1 + k_2\vec{n}_2\vec{n}_2), \tag{3}$$

and a rotating elastic bedding given by

$$\vec{\vec{K}}_{\text{rot}} = (c_1\vec{e}_1\vec{e}_1 + c_2\vec{e}_2\vec{e}_2). \tag{4}$$

The constants c_1 and c_2 can easily be calculated from EI_1, EI_2 by considering the equations of a massless elastic beam with appropriate boundary conditions. At the mounting of the shaft for the restoring force $\vec{F}$ we have

$$\vec{r} = -\vec{\vec{K}}^{-1}\vec{F} = -\left(\vec{\vec{K}}_{\text{stat}}^{-1} + \vec{\vec{K}}_{\text{rot}}^{-1}\right)\vec{F}, \tag{5}$$

so that the resulting stiffness tensor follows as

$$\vec{\vec{K}} = \left(\vec{\vec{K}}_{\text{stat}}^{-1} + \vec{\vec{K}}_{\text{rot}}^{-1}\right)^{-1}. \tag{6}$$

The forces in the right-hand side of Eq. (1) are due to internal and external damping given by

$$\vec{F}_d = -d_i(\dot{\vec{r}} - \Omega\vec{n}_3 \times \vec{r}) - d_e\dot{\vec{r}} \tag{7}$$

and contact forces which read

$$\vec{F}_c = -k_c(\vec{r} \cdot \vec{n}_1 - \mu\vec{r} \cdot \vec{n}_2)\vec{n}_1 - k_c(\vec{r} \cdot \vec{n}_2 + \mu\vec{r} \cdot \vec{n}_1)\vec{n}_2 - d_c\dot{\vec{r}}, \tag{8}$$

where μ is the coefficient of dry friction as used in Coulomb's law. The parameters k_c and d_c are the stiffness and damping coefficients of the contact. Note that we consider rotation with $\Omega > 0$; otherwise, one must take the dry friction coefficient μ with opposite sign in Eq. (8). The model can be seen as an extension of an asymmetric rotor studied in [11] in the sense that damping and contact forces are considered. However, in our case we concentrate on stability behavior of the homogeneous equations of motion and therefore neglect any kind of unbalance.

So far all the expressions presented were invariant of coordinates. In order to make computations we have to specify the degrees of freedom for our problem. A convenient formulation will be achieved by defining the coordinates with respect to the inertial frame, i.e.,

$$\vec{r} = q_1\vec{n}_1 + q_2\vec{n}_2. \tag{9}$$

We introduce the following dimensionless variables and parameters:

$$\tilde{t} = \omega t, \quad \tilde{\Omega} = \frac{\Omega}{\omega}, \quad \tilde{k}_1 = \frac{k_1}{m\omega^2}, \quad \tilde{k}_2 = \frac{k_2}{m\omega^2}, \quad \tilde{k}_c = \frac{k_c}{m\omega^2},$$
$$\tilde{c}_1 = \frac{c_1}{m\omega^2}, \quad \tilde{c}_2 = \frac{c_2}{m\omega^2}, \quad \tilde{\xi} = \frac{\mu k_c}{m\omega^2}, \quad \tilde{\delta}_i = \frac{d_i}{m\omega}, \quad \tilde{\delta}_e = \frac{d_e + d_c}{m\omega}, \tag{10}$$

where the characteristic frequency ω is defined as

$$\omega^2 = \frac{ck}{m(c+k)} + \frac{k_c}{m}, \quad k = \frac{k_1 + k_2}{2}, \quad c = \frac{c_1 + c_2}{2}. \tag{11}$$

Dimensionless equations are obtained from Eq. (1) in matrix form (with the tildes omitted) as

$$\ddot{\mathbf{q}} + (\delta_i + \delta_e)\dot{\mathbf{q}} + \mathbf{K}(t)\mathbf{q} = \mathbf{0}, \quad \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{12}$$

Here

$$\mathbf{K} = (\mathbf{K}_{\text{stat}}^{-1} + \mathbf{K}_{\text{rot}}^{-1})^{-1} + k_c\mathbf{I} + (\delta_i\Omega - \xi)\mathbf{N} \tag{13}$$

with

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{K}_{\text{stat}} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \tag{14}$$

$$\mathbf{K}_{\text{rot}} = \begin{pmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{pmatrix} \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{pmatrix}, \quad \Phi = \Omega t. \tag{15}$$

It is easy to see that the coefficients in Eq. (12) are periodic with respect to time with period $T = \pi/\Omega$. Note that the matrix $\mathbf{K}$ is also periodic if we write equations of motion in the rotating frame, which means that there is no easy formulation with constant coefficients. This is generally the case, when both stationary and rotating restoring stiffnesses are asymmetric. This phenomenon becomes particularly technically relevant in rotor problems with contact.

The goal is now to study the system with analytic perturbation theory and to calculate first-order approximations to the stability boundaries. Therefore, we consider all terms, which break the symmetry of the system, and nonconservative forces as perturbations, and linearize coefficients in Eq. (12) with respect to parameters about $c_1 = c_2 = c$ and $k_1 = k_2 = k$. This yields

$$\mathbf{K} = \mathbf{I} + \Delta\mathbf{K}, \quad \Delta\mathbf{K} = \Delta_c \begin{pmatrix} -\cos(2\Omega t) & -\sin(2\Omega t) \\ -\sin(2\Omega t) & \cos(2\Omega t) \end{pmatrix} + \begin{pmatrix} -\Delta_k & \delta_i\Omega - \xi \\ \xi - \delta_i\Omega & \Delta_k \end{pmatrix} \tag{16}$$

with

$$\Delta_c = \frac{c_2 - c_1}{2(1 + c/k)^2}, \quad \Delta_k = \frac{k_2 - k_1}{2(1 + k/c)^2}, \tag{17}$$

where the higher-order terms $o(\Delta_k, \Delta_c)$ were neglected. Note that ω from Eq. (11) is the eigenfrequency of the unperturbed system, which in dimensionless formulation reduces to unity, $ck/(c+k) + k_c = 1$. The final expression (16) depends on the dimensionless stiffness parameters $k_{1,2}$ and $c_{1,2}$ through their relations given in Eq. (17), while the contact stiffness k_c appears only in the dimensional expressions (10) and (11).

Remark. When the contact is rotating with the rotor like, for example, in a centrifugal clutch [16], the equations have similar structure. Note that an asymmetry in the contact leads to breaking of axial symmetry in the damping. Such asymmetry is not considered here, since in a similar analysis in [17] no qualitatively different results were observed.

3. Eigenvalues in the context of Floquet theory

Let us write system (12) in the form of a first-order system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \tag{18}$$

with

$$\mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}(t) & -(\delta_i + \delta_e)\mathbf{I} \end{pmatrix}, \tag{19}$$

where $\mathbf{0}$ is the 2×2 zero matrix. When asymmetry is small, in the first-order approximation we have

$$\mathbf{A}(t) = \mathbf{A}_0 + \mathbf{A}_1(t), \tag{20}$$

$$\mathbf{A}_0 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{A}_1(t) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -\Delta\mathbf{K}(t) & -(\delta_i + \delta_e)\mathbf{I} \end{pmatrix}, \tag{21}$$

where $\mathbf{A}_0$ describes the unperturbed system and $\mathbf{A}_1$ is a small perturbation, see Eq. (16).

For stability analysis of the time-periodic system (18), one has to compute the Floquet matrix $\mathbf{F} = \mathbf{X}(T)$, where $\mathbf{X}(t)$ is the 4×4 matrix solving the system

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I} \tag{22}$$

with the 4×4 identity matrix as the initial condition. The system is asymptotically stable if $|\rho| < 1$ for all eigenvalues (also called Floquet multipliers) ρ of the matrix $\mathbf{F}$, see [18, Section 9.2]. If there is a multiplier with $|\rho| > 1$, the system is unstable. When $|\rho| \le 1$ for all multipliers, then the system is stable if and only if all the multipliers with $|\rho| = 1$ are simple or semi-simple. Recall that semi-simple means that the eigenvalue is multiple but possesses as many linearly independent eigenvectors as its algebraic multiplicity.

We remark that the Floquet matrix determines the transfer map for the system over half a period of rotation $T = \pi/\Omega$, i.e., when the rotor rotates by 180°. This implies that computing a similar Floquet matrix for equations of motion written in the rotating frame yields the same matrix but with a different sign. After two periods, i.e., a full rotation of the rotor, the corresponding matrix $\mathbf{F}^2$ is identical for rotating and stationary coordinates.

The unperturbed system matrix $\mathbf{A}_0$ is time independent. Its eigenvalues λ with right eigenvectors $\mathbf{u}$ and left eigenvectors $\mathbf{v}$ are determined by the

$$\mathbf{A}_0\mathbf{u} = \lambda\mathbf{u}, \quad \mathbf{v}^{\mathrm{T}}\mathbf{A}_0 = \lambda\mathbf{v}^{\mathrm{T}}, \tag{23}$$

where $\mathbf{v}^{\mathrm{T}}$ is the transposed vector. Using the explicit form of the matrix $\mathbf{A}_0$ from Eq. (21) we find the eigenvalues and eigenvectors as

$$\lambda_{\pm} = \pm\mathrm{i}, \quad \mathbf{u}_{\pm 1} = \begin{pmatrix} 1 \\ 0 \\ \lambda_{\pm} \\ 0 \end{pmatrix}, \quad \mathbf{u}_{\pm 2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \lambda_{\pm} \end{pmatrix}, \quad \mathbf{v}_{\pm 1} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2\lambda_{\pm}} \\ 0 \end{pmatrix}, \quad \mathbf{v}_{\pm 2} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2\lambda_{\pm}} \end{pmatrix}. \tag{24}$$

Eigenvectors with the same sign correspond to the same eigenvalue. Thus, the eigenvalues $\lambda_+ = \mathrm{i}$ and $\lambda_- = -\mathrm{i}$ are double and semi-simple. Note that the eigenvectors satisfy the normalization conditions

$$\mathbf{v}_{\sigma j}^{\mathrm{T}}\mathbf{u}_{\sigma' j'} = \delta_j^{j'}\delta_\sigma^{\sigma'}, \quad j, j' = 1, 2, \; \sigma, \sigma' = \pm, \tag{25}$$

where $\delta_j^{j'}$ is the Kronecker delta.

Since the matrix $\mathbf{A}_0$ is time-independent, we find the matrices $\mathbf{X}(t)$ and $\mathbf{F}$ for the unperturbed system in the form of matrix exponentials as

$$\mathbf{X}_0(t) = \exp(t\mathbf{A}_0), \quad \mathbf{F}_0 = \exp(T\mathbf{A}_0) = \exp\left(\frac{\pi}{\Omega}\mathbf{A}_0\right). \tag{26}$$

Multipliers (eigenvalues) of the Floquet matrix $\mathbf{F}_0$ from Eq. (26) are found as

$$\rho_{\pm} = \exp(T\lambda_{\pm}) = \exp(\pm\mathrm{i}\pi\Omega^{-1}), \tag{27}$$

where $\lambda_{\pm} = \pm\mathrm{i}$ are eigenvalues of $\mathbf{A}_0$, see Eq. (24). The eigenvectors remain the same [25,18], so that

$$\mathbf{F}_0\mathbf{u}_{\pm j} = \rho_{\pm}\mathbf{u}_{\pm j}, \quad \mathbf{v}_{\pm j}^{\mathrm{T}}\mathbf{F}_0 = \rho_{\pm}\mathbf{v}_{\pm j}^{\mathrm{T}}. \tag{28}$$

If $\rho_+ = \rho_-$ then the resulting multiplier is semi-simple and its multiplicity is equal to four. This occurs at the resonant rotation speeds, when

$$\Omega = 1/n, \quad \rho_+ = \rho_- = (-1)^n, \; n = 1, 2, \ldots \tag{29}$$

At non-resonant values of Ω, we have $\rho_+ \neq \rho_-$ and the multipliers are double and semi-simple. Similar to Eq. (28), for the matrix $\mathbf{X}_0(t)$ from Eq. (26) and its inverse, we have [25,18]

$$\mathbf{X}_0(t)\mathbf{u}_{\pm j} = e^{\lambda_{\pm}t}\mathbf{u}_{\pm j} = e^{\pm\mathrm{i}t}\mathbf{u}_{\pm j}, \quad \mathbf{v}_{\pm j}^{\mathrm{T}}\mathbf{X}_0^{-1}(t) = e^{-\lambda_{\pm}t}\mathbf{v}_{\pm j}^{\mathrm{T}} = e^{\mp\mathrm{i}t}\mathbf{v}_{\pm j}^{\mathrm{T}}. \tag{30}$$

4. Perturbation of double and quadruple multipliers

Let us now consider the perturbed system with the time-dependent matrix operator (20). We will also assume a small perturbation T_1 of the rotation period,

$$T = T_0 + T_1, \tag{31}$$

caused by the variation of rotation speed. The Floquet matrix is a smooth function of parameters. We can write the perturbed Floquet matrix keeping only the first-order (linear) correction term as

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1, \tag{32}$$

where [18, Section 9.3]

$$\mathbf{F}_1 = \mathbf{F}_0\int_0^{T_0}\mathbf{X}_0^{-1}\mathbf{A}_1\mathbf{X}_0\,\mathrm{d}t + T_1\mathbf{A}_0\mathbf{F}_0. \tag{33}$$

The semi-simple multiplier ρ_+ of the matrix $\mathbf{F}_0$ is given in Eq. (27), and its small perturbation is denoted as

$$\rho = \rho_+ + \rho_1 + o(\rho_1), \quad \rho_1 = \rho_+\eta, \tag{34}$$

where ρ_1 is the leading correction term and η is introduced for further convenience. When $\rho_+ \neq \rho_-$, analogous expression for the perturbed multiplier ρ_- is obtained from Eq. (34) by complex conjugation; for this reason we will only consider ρ_+ from now on.

For absolute values of perturbed multipliers (34), where $|\rho_+|=1$, we have

$$|\rho| = |1+\eta+o(\eta)| = 1+\mathrm{Re}(\eta)+o(\eta). \tag{35}$$

For asymptotic stability, we need $|\rho|<1$ and, hence,

$$\mathrm{Re}\,\eta < 0. \tag{36}$$

The imaginary part of η only affects higher-order terms in Eq. (35) and, thus, it does not contribute to the first-order stability condition (36). When the contact and dissipative forces are absent, i.e., $\xi=\delta_i=\delta_e=0$, the system has Hamiltonian structure and the multipliers possess the symmetry $\rho\mapsto 1/\rho$ [26]. In this case a stable system must have all multipliers on the unit circle $|\rho|=1$. Hence, all values of η in Eq. (35) are purely imaginary for a stabilizing perturbation [18].

In the non-resonant case $\Omega \neq 1/n$, there are two values of ρ_1 describing splitting of the double multiplier ρ_+. These two values are determined as the eigenvalues of the 2×2 matrix [18, Section 9.7]

$$\det\begin{pmatrix} \mathbf{v}_{+1}^T\mathbf{F}_1\mathbf{u}_{+1}-\rho_1 & \mathbf{v}_{+1}^T\mathbf{F}_1\mathbf{u}_{+2} \\ \mathbf{v}_{+2}^T\mathbf{F}_1\mathbf{u}_{+1} & \mathbf{v}_{+2}^T\mathbf{F}_1\mathbf{u}_{+2}-\rho_1 \end{pmatrix} = 0. \tag{37}$$

Substituting Eq. (33) into Eq. (37) and using Eqs. (23), (25), (28), (30) for the first element yield

$$\mathbf{v}_{+1}^T\mathbf{F}_1\mathbf{u}_{+1} = \mathbf{v}_{+1}^T\mathbf{F}_0\int_0^{T_0}\mathbf{X}_0^{-1}\mathbf{A}_1\mathbf{X}_0\,\mathrm{d}t\,\mathbf{u}_{+1} + T_1\mathbf{v}_{+1}^T\mathbf{A}_0\mathbf{F}_0\mathbf{u}_{+1} = \rho_+\left(\int_0^{T_0}\mathbf{v}_{+1}^T\mathbf{A}_1\mathbf{u}_{+1}\,\mathrm{d}t + \mathrm{i}T_1\right). \tag{38}$$

Performing a similar computation for other elements and omitting the common factor ρ_+, we reduce Eq. (37) to the form

$$\det(\mathbf{G}_0-(\eta-\mathrm{i}T_1)\mathbf{I})=0, \tag{39}$$

where elements of the 2×2 matrix $\mathbf{G}_0=[g_{jk}^0]$ are given by

$$g_{jk}^0 = \int_0^{T_0}\mathbf{v}_{+j}^T\mathbf{A}_1\mathbf{u}_{+k}\,\mathrm{d}t, \quad j,k=1,2. \tag{40}$$

We see that the bifurcation of a double multiplier is determined by the matrix $\mathbf{G}_0$, which represents the mean value of $\mathbf{A}_1$ written in an appropriate coordinate basis, as discovered earlier in [15,16].

In the resonant case (29), we have $T_0=\pi n$ and there is a single semi-simple multiplier of multiplicity four. Four values of ρ_1 in the perturbation expression (34) are given by the eigenvalues of the 4×4 matrix [18, Section 2.10]

$$\det\begin{pmatrix} \mathbf{v}_{+1}^T\mathbf{F}_1\mathbf{u}_{+1}-\rho_1 & \mathbf{v}_{+1}^T\mathbf{F}_1\mathbf{u}_{+2} & \mathbf{v}_{+1}^T\mathbf{F}_1\mathbf{u}_{-1} & \mathbf{v}_{+1}^T\mathbf{F}_1\mathbf{u}_{-2} \\ \mathbf{v}_{+2}^T\mathbf{F}_1\mathbf{u}_{+1} & \mathbf{v}_{+2}^T\mathbf{F}_1\mathbf{u}_{+2}-\rho_1 & \mathbf{v}_{+2}^T\mathbf{F}_1\mathbf{u}_{-1} & \mathbf{v}_{+2}^T\mathbf{F}_1\mathbf{u}_{-2} \\ \mathbf{v}_{-1}^T\mathbf{F}_1\mathbf{u}_{+1} & \mathbf{v}_{-1}^T\mathbf{F}_1\mathbf{u}_{+2} & \mathbf{v}_{-1}^T\mathbf{F}_1\mathbf{u}_{-1}-\rho_1 & \mathbf{v}_{-1}^T\mathbf{F}_1\mathbf{u}_{-2} \\ \mathbf{v}_{-2}^T\mathbf{F}_1\mathbf{u}_{+1} & \mathbf{v}_{-2}^T\mathbf{F}_1\mathbf{u}_{+2} & \mathbf{v}_{-2}^T\mathbf{F}_1\mathbf{u}_{-1} & \mathbf{v}_{-2}^T\mathbf{F}_1\mathbf{u}_{-2}-\rho_1 \end{pmatrix} = 0. \tag{41}$$

Calculations analogous to Eq. (38) with the complex conjugate relation of eigenvectors $\mathbf{u}_{-k}=\overline{\mathbf{u}_{+k}}$ and $\mathbf{v}_{-j}=\overline{\mathbf{v}_{+j}}$ following from Eq. (24) yield

$$\det\begin{pmatrix} \mathbf{G}_0-(\eta-\mathrm{i}T_1)\mathbf{I} & \mathbf{G}_1 \\ \overline{\mathbf{G}_1} & \overline{\mathbf{G}_0}-(\eta+\mathrm{i}T_1)\mathbf{I} \end{pmatrix} = 0, \tag{42}$$

where $\mathbf{G}_0$ is given by Eq. (40) and elements of the 2×2 matrix $\mathbf{G}_1=[g_{jk}^1]$ are

$$g_{jk}^1 = \int_0^{T_0}\mathbf{v}_{+j}^T\mathbf{A}_1\mathbf{u}_{-k}e^{-\mathrm{i}2t}\,\mathrm{d}t. \tag{43}$$

Here the factor $e^{-\mathrm{i}2t}$ appears because $\mathbf{v}_{+j}^T$ and $\mathbf{u}_{-k}$ yield the same multiplier $e^{-\mathrm{i}t}$ in Eq. (30). We see that the bifurcation of the quadruple multiplier is determined by the mean value and the specific Fourier coefficient of the periodic matrix $\mathbf{A}_1(t)$.

Note that the theory given in this section is rather general. We expect that it can be extended to general rotating systems with axial symmetry.

5. Non-resonant stability conditions

Let us apply the formulas of the previous section in the non-resonant case with the matrix $\mathbf{A}_1$ from Eq. (21). Then, expressions (40) with (24), (21) and (16) yield

$$\mathbf{G}_0 = -\frac{T_0}{2\mathrm{i}}\left(\langle\Delta\mathbf{K}\rangle+(\delta_i+\delta_e)\mathrm{i}\mathbf{I}\right), \tag{44}$$

where

$$\langle\Delta\mathbf{K}\rangle = \begin{pmatrix} -\Delta_k & \delta_i\Omega-\xi \\ \xi-\delta_i\Omega & \Delta_k \end{pmatrix} \tag{45}$$

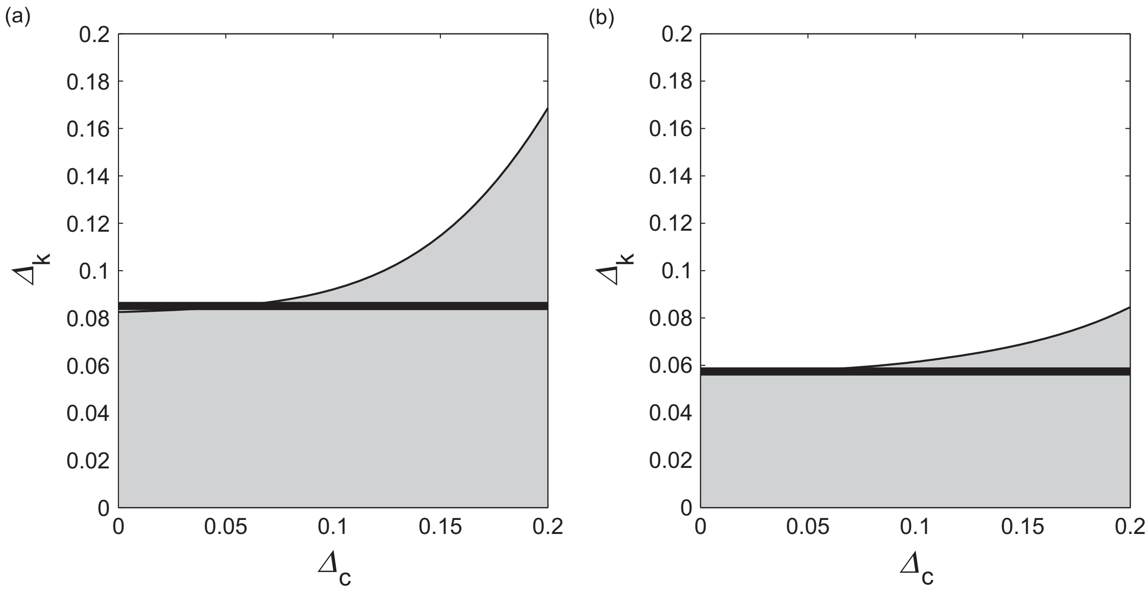

Fig. 2. Stability boundaries for (a) subcritical rotation speed $\Omega = 0.3$ and (b) supercritical rotation speed $\Omega = 1.5$. Gray regions show the numerical instability regions and horizontal bold lines denote the theoretical first-order approximations of the stability boundary.

is the mean value of the matrix in Eq. (16). In the non-resonant case the period perturbation T_1 leads to purely imaginary change of η in Eq. (39) and, hence, it has no effect on the stability condition (36). For this reason, we can take $T_1 = 0$ and write the characteristic equation in Eq. (39) as

$$\eta^2 + T_0(\delta_i + \delta_e)\eta + T_0^2\left[\Delta_k^2 + (\delta_i + \delta_e)^2 - (\xi - \delta_i\Omega)^2\right]/4 = 0. \tag{46}$$

For asymptotic stability, condition (36) must be satisfied for both roots η of Eq. (46). The Routh–Hurwitz criterion yields

$$\delta_i + \delta_e > 0, \quad \Delta_k^2 + (\delta_i + \delta_e)^2 > (\xi - \delta_i\Omega)^2. \tag{47}$$

The first condition is usually fulfilled, since the damping coefficients are nonnegative. When $\delta_i > 0$, the second condition provides the stability interval for rotation speeds (away from resonance zones considered in the next section) as

$$\max(0, \Omega_-) < \Omega < \Omega_+, \quad \Omega_\pm = \frac{\xi}{\delta_i} \pm \sqrt{\frac{\Delta_k^2}{\delta_i^2} + \left(1 + \frac{\delta_e}{\delta_i}\right)^2}. \tag{48}$$

Note that, in the absence of internal damping, $\delta_i = 0$, conditions (47) reduce to the ones found in [17].

We see from Eq. (47) that the system is stabilized by the static stiffness asymmetry Δ_k and external damping δ_e. The effect of internal damping δ_i depends on the values of other parameters and can only destabilize the system in the supercritical range $\Omega > 1$, which is well-known for rotors described by systems with constant coefficients, see, e.g., [9]. In particular, internal damping acts against the contact force and compensates it when $\delta_i\Omega = \xi$. The asymmetry of the rotor stiffness Δ_c has no effect on stability in the first approximation, though it may influence higher order correction terms.

For computations, let us consider a numerical example with the dimensional parameters (arbitrary units)

$$m = 1, \quad k = c = 1, \quad k_c = 0.5, \quad \mu = 0.2, \quad d_i = d_e + d_c = 0.02, \tag{49}$$

which yield $\omega = 1$ in Eq. (11). For dimensionless parameters (10) and (17) we have

$$\xi = 0.1, \quad \delta_i = \delta_e = 0.02, \quad \Delta_k = \frac{k_2 - k_1}{8k}, \quad \Delta_c = \frac{c_2 - c_1}{8c}. \tag{50}$$

The range $|\Delta_k| < 1/4$ and $|\Delta_c| < 1/4$ for the asymmetry parameters follows from the positivity of all stiffnesses. The symmetric system with $\Delta_k = \Delta_c = 0$ is stable in the supercritical interval of rotation speeds, $3 < \Omega < 7$, obtained from the first-order condition (48); numerical computations of the Floquet matrix confirm this result with the accuracy of about 0.03 percent. The instability should be attributed to the nonconservative follower force given by the dry friction parameter μ (dimensionless parameter ξ).

Fig. 2 shows stability diagrams for the fixed rotation speeds $\Omega = 0.3$ and $\Omega = 1.5$. The first-order stability condition (47) provides the critical value of static asymmetry Δ_k^{cr}, such that the system is stabilized for

$$|\Delta_k| > \Delta_k^{cr} = \sqrt{(\xi - \delta_i\Omega)^2 - (\delta_i + \delta_e)^2}. \tag{51}$$

These values are shown in Fig. 2 by the horizontal bold lines. Gray regions show the instability domain of system (12)–(15) obtained by checking numerically the asymptotic stability condition $|\rho|<1$ for eigenvalues of the Floquet matrix for a dense grid in the parameter space. This requires numerical computation of the Floquet matrix at all grid points, which can be effectively done using parallel computing. Note that the deviation of the theoretical result from the exact stability boundary is the higher order effect and becomes essential with increasing Δ_c (recall that $|\Delta_c|<1/4$). For both cases in Fig. 2, the higher order effects of rotating asymmetry Δ_c favor destabilization of the system.

In the case of no contact forces and no damping, $\xi=\delta_i=\delta_e=0$, Eq. (46) yields

$$\eta=\pm \mathrm{i}\frac{T_0\Delta_k}{2}. \tag{52}$$

As explained in Section 4, purely imaginary values of η imply that the two multipliers split along the unit circle $|\rho|=1$. Therefore, the system remains stable for small asymmetry of static and rotating stiffnesses at non-resonant rotation speeds.

6. Trefoil instability zones near resonant rotation speeds

Let us consider rotation speeds $\Omega=\Omega_0+\Delta_\Omega$ near the resonant value $\Omega_0=1$, at which the unperturbed system has the semi-simple multiplier $\rho_+=\rho_-=-1$ of multiplicity four. Bifurcation of this quadruple multiplier into four simple multipliers under a small change of parameters is described by Eq. (34), where four values of η are given by roots of the characteristic equation (42). The block matrices $\mathbf{G}_0$ on the diagonal are given by expressions (44) and (45). The off-diagonal blocks $\mathbf{G}_1$ can be calculated using expressions (43) with (24), (21) and (16). Both $\mathbf{G}_0$ and $\mathbf{G}_1$ are taken for the unperturbed system with $\Omega_0=1$ and $T_0=\pi/\Omega_0=\pi$, which yields

$$\mathbf{G}_0=-\frac{\pi}{2\mathrm{i}}\begin{pmatrix}-\Delta_k+(\delta_i+\delta_e)\mathrm{i} & \delta_i-\xi\\ \xi-\delta_i & \Delta_k+(\delta_i+\delta_e)\mathrm{i}\end{pmatrix},\quad \mathbf{G}_1=-\frac{\pi\Delta_c}{4\mathrm{i}}\begin{pmatrix}-1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}. \tag{53}$$

The period of perturbed system $T=\pi/\Omega$ is given up to small terms $o(\Delta_\Omega)$ by Eq. (31) with

$$T_0=\frac{\pi}{\Omega_0}=\pi,\quad T_1=-\frac{\pi\Delta_\Omega}{\Omega_0^2}=-\pi\Delta_\Omega. \tag{54}$$

Substituting Eqs. (53) and (54) into Eq. (42) and multiplying by $(2/\pi)^4$ yield

$$\tilde{\eta}^4+a_2\tilde{\eta}^2+a_1\tilde{\eta}+a_0=0,\quad \tilde{\eta}=\frac{2\eta}{\pi}+\delta_i+\delta_e, \tag{55}$$

where the coefficients are

$$a_0=\left[4\Delta_\Omega^2-\Delta_k^2+(\xi-\delta_i)^2\right]^2-\Delta_c^2\left[4\Delta_\Omega^2+(\xi-\delta_i)^2\right], \tag{56}$$

$$a_1=2\Delta_c^2(\xi-\delta_i), \tag{57}$$

$$a_2=8\Delta_\Omega^2+2\Delta_k^2-\Delta_c^2-2(\xi-\delta_i)^2. \tag{58}$$

As we showed above, the first-order asymptotic stability condition is given by Eq. (36). Using the second expression in Eq. (55), we write this condition as

$$\mathrm{Re}\,\tilde{\eta}<\delta_i+\delta_e \tag{59}$$

for all four roots $\tilde{\eta}$.

Note that exactly at the resonance, $\Delta_\Omega=0$, the polynomial in Eq. (55) factorizes and yields the two quadratic equations

$$\tilde{\eta}^2+\sigma\Delta_c\tilde{\eta}+\Delta_k^2-\sigma\Delta_c(\xi-\delta_i)-(\xi-\delta_i)^2=0,\quad \sigma=\pm 1. \tag{60}$$

6.1. Stability of system with no contact force and no damping

Let us now analyze the stability regions in the parameter space. In the case of no damping and no contact forces, i.e., $\delta_i=\delta_e=\xi=0$, the polynomial (55) reduces to

$$\tilde{\eta}^4+b_1\tilde{\eta}^2+b_0=0,\quad \tilde{\eta}=\frac{2\eta}{\pi}, \tag{61}$$

with the coefficients

$$b_0=(4\Delta_\Omega^2-\Delta_k^2)^2-4\Delta_\Omega^2\Delta_c^2,\quad b_1=8\Delta_\Omega^2+2\Delta_k^2-\Delta_c^2. \tag{62}$$

As we mentioned in Section 4, for stability of the system without contact forces and damping, all roots $\tilde{\eta}$ must be purely imaginary. Thus, $\tilde{\eta}^2$ must be real and negative. In terms of the polynomial coefficients this stability condition reads

$$b_0 > 0, \quad b_1 > 0, \quad D = b_1^2 - 4b_0 > 0, \tag{63}$$

where the first two inequalities are the Routh–Hurwitz conditions ensuring that Re $\tilde{\eta}^2 < 0$, and the last condition leads to real roots $\tilde{\eta}^2$. We use strict inequalities in Eq. (63) determining interior of the stability domain. Stability at the boundary of this domain has minor importance for our problem, and it depends on the Jordan structure of multipliers [18].

Using Eq. (62), we write inequalities (63) explicitly in terms of system parameters as

$$\begin{gathered}(4\Delta_\Omega^2 - \Delta_k^2)^2 > 4\Delta_\Omega^2\Delta_c^2, \quad 8\Delta_\Omega^2 + 2\Delta_k^2 > \Delta_c^2, \\ 64\Delta_k^2\Delta_\Omega^2 + \Delta_c^4 > 4\Delta_k^2\Delta_c^2.\end{gathered} \tag{64}$$

Fig. 3 shows the instability domain in the parameter space $(\Delta_\Omega, \Delta_k, \Delta_c)$ determined as an exterior of the region given by Eq. (64). This domain has a distinctive trefoil structure, which originates from the bifurcation of the quadruple multiplier and, therefore, should be typical for rotors with static and rotating asymmetries.

It is easy to see that the system is stable for the static asymmetry $\Delta_k \neq 0$ in the absence of the rotating asymmetry $\Delta_c = 0$. In the opposite case of no static asymmetry $\Delta_k = 0$ and nonzero rotating asymmetry $\Delta_c \neq 0$, the stability conditions reduce to

$$2|\Delta_\Omega| > |\Delta_c|. \tag{65}$$

Therefore, the system is destabilized in a small interval of rotation speeds near the resonant value in agreement with well-known results, e.g., [1,9].

Our new result is the stability region under the combined action of static and rotation asymmetries given in Eq. (64). Fig. 4 shows the instability zones in the parameter plane for different ratios of Δ_k and Δ_c (two sections of the domain in Fig. 3). The first-order analytical results are presented and confirmed numerically by integrating system (22) using the Runge–Kutta method and checking the stability condition for resulting multipliers. For large values of asymmetries,

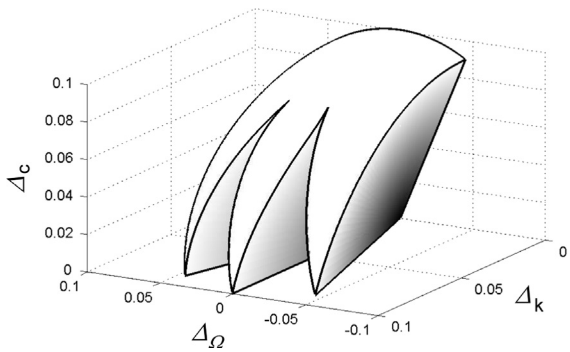

Fig. 3. Trefoil instability domain in the space of three parameters: detuning of the rotation speed Δ_Ω, static asymmetry Δ_k and rotating asymmetry Δ_c. For the visualization purpose, we cut the instability region by a sphere. This region extends to negative Δ_k or Δ_c by symmetry.

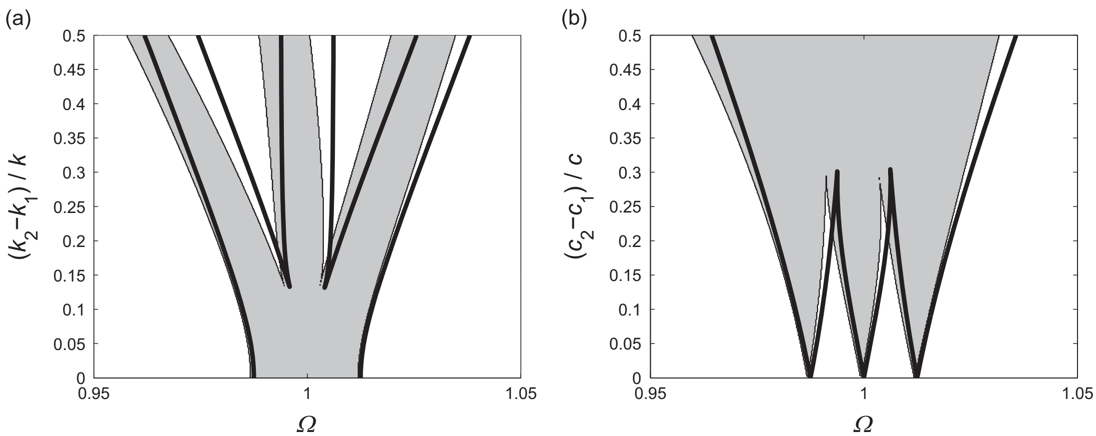

Fig. 4. Trefoil instability zones in parameter space near the resonance point $\Omega_0 = 1$ for a rotor with no contact force and no dissipation. Bold lines show the first-order analytic approximation of the stability boundary and the gray instability regions represent the results of numerical simulations. (a) Fixed $(c_2 - c_1)/c = 8\Delta_c = 0.2$ with variable Ω and $(k_2 - k_1)/k = 8\Delta_k$. (b) Fixed $(k_2 - k_1)/k = 8\Delta_k = 0.2$ with variable Ω and $(c_2 - c_1)/c = 8\Delta_c$.

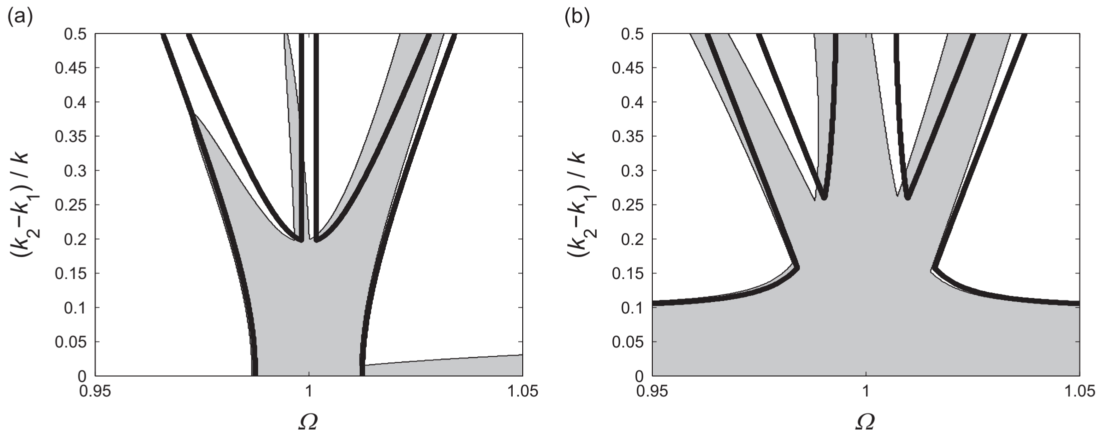

Fig. 5. Resonant instability zones in parameter space near the resonance point $\Omega_0 = 1$ for a rotor with nonconservative forces. The bold curves show the first-order analytic approximation of the stability boundary and the gray instability regions represent the results of numerical simulations. The rotating asymmetry is chosen to be $(c_2 - c_1)/c = 8\Delta_c = 0.2$ as in Fig. 4a. (a) Effect of internal damping $\delta_i = 0.012$ when $\xi = \delta_e = 0$. (b) Joint effect of internal and external damping $\delta_i = \delta_e = 0.002$ at the prevailing contact force $\xi = 0.015$.

deviations from the first-order conditions become significant, though the stability regions retain their qualitative form. The system has the general tendency to increase the unstable frequency interval with the increase of rotor asymmetry, which is similar to the classical rotor with rigid bedding [1].

We conclude that the trefoil shape of the instability region results from the interaction of four modes, which form a quadruple multiplier at the resonant point. One can see the drastic change in the structure of the instability region as compared to the system with only rotating asymmetry, when a single tongue of instability appears. This difference is attributed exclusively to intrinsic periodicity of system coefficients or, mechanically, to the combined effect of static and rotating asymmetries of the rotor system.

6.2. Stability conditions for a rotor with nonconservative forces

Now let us return to the general case with nonzero contact force and damping. In this case stability is determined by the condition (59) for the roots $\tilde{\eta}$ given by Eq. (55). The value of $\tilde{\eta}$ in Eqs. (55) and (58) does not depend on δ_e. Hence, adding only external damping with no internal damping and no contact forces increases the stability region, which is an expected result.

Setting damping to zero $\delta_i = \delta_e = 0$ and assuming $\xi > 0$ destabilize the system in the presence of rotating asymmetry $\Delta_c \neq 0$. One can check this by using the Routh–Hurwitz criterion, which shows instability for a polynomial of fourth order (55) with $a_3 = 0$ and $a_1 \neq 0$.

The effect of internal damping with no contact force and no external damping depends nonlinearly on other perturbation parameters. Fig. 5a shows a change of the instability region in Fig. 4a for small internal damping $\delta_i = 0.012$ and $\xi = \delta_e = 0$. We see that the resonant zone is weakly affected, while a small destabilization region appears in supercritical range $\Delta_\Omega > 0$, in agreement with the non-resonant stability condition (47).

Exactly at the resonance, i.e., when $\Delta_\Omega = 0$, the four values of $\tilde{\eta}$ are found from Eq. (60). Writing the polynomial in Eq. (60) in terms of η and applying the Routh–Hurwitz criterion for both signs $\sigma = \pm 1$, we find the stability conditions as

$$|\Delta_c| < 2(\delta_i + \delta_e), \quad (\delta_i + \delta_e)^2 + \Delta_k^2 - (\xi - \delta_i)^2 > |\Delta_c(\xi - 2\delta_i - \delta_e)|. \tag{66}$$

For nonzero Δ_Ω, explicit conditions of asymptotic stability can be written in the form of Routh–Hurwitz inequalities

$$\alpha_2\alpha_3 > \alpha_1, \quad \alpha_1\alpha_2\alpha_3 > \alpha_1^2 + \alpha_0\alpha_3^2, \quad \alpha_j > 0,\ j = 0, \dots, 3, \tag{67}$$

where α_j are the coefficients of polynomial (55) written in powers of η instead of $\tilde{\eta}$. In terms of the coefficients (56)–(58), they yield

$$\begin{gathered} 20(\delta_i + \delta_e)^3 + 2a_2(\delta_i + \delta_e) > a_1, \\ 4(\delta_i + \delta_e)^2 \left[a_2 + 4(\delta_i + \delta_e)^2\right]^2 > a_1^2 + 16a_0(\delta_i + \delta_e)^2, \\ (\delta_i + \delta_e)^4 + a_2(\delta_i + \delta_e)^2 + a_1(\delta_i + \delta_e) + a_0 > 0, \\ 4(\delta_i + \delta_e)^3 + 2a_2(\delta_i + \delta_e) + a_1 > 0, \\ 6(\delta_i + \delta_e)^2 + a_2 > 0, \\ \delta_i + \delta_e > 0. \end{gathered} \tag{68}$$

Despite rather cumbersome form, stability conditions (68) have the advantage that they do not require integration of time-dependent equations of motion. We expect that the first-order stability conditions for higher dimensional models of a rotor have a similar form, resulting from the mathematical structure of bifurcation of quadruple semi-simple eigenvalue. Fig. 5b shows an example of the joint effect of contact force and dissipation on the stability region in the parameter plane (Ω, Δ_k), which can be compared to Fig. 4a.

6.3. Secondary resonant zones

Finally, let us consider secondary resonances in the neighborhood of the rotation speeds $\Omega_0 = 1/n$ for $n = 2, 3, \ldots$, see Eq. (29). Expressions (54) for the system period and its perturbation become

$$T_0 = n\pi, \quad T_1 = -n^2\pi\Delta_\Omega. \tag{69}$$

Using Eqs. (21) and (16), one can see that the matrix $\mathbf{A}_1$ contains only constant terms and terms proportional to $\cos(2t/n)$ and $\sin(2t/n)$. This yields a zero matrix $\mathbf{G}_1 = 0$ given by the integral (43) for $n \geq 2$. The matrix $\mathbf{G}_0$ is determined by Eqs. (44) and (45) as

$$\mathbf{G}_0 = -\frac{n\pi}{2\mathrm{i}}\begin{pmatrix} -\Delta_k + (\delta_i + \delta_e)\mathrm{i} & \delta_i/n - \xi \\ \xi - \delta_i/n & \Delta_k + (\delta_i + \delta_e)\mathrm{i} \end{pmatrix}. \tag{70}$$

In this case Eq. (42) reduces to a pair of quadratic equations represented as

$$\det\left[\mathbf{G}_0 - (\eta - \mathrm{i}T_1)\mathbf{I}\right] = 0 \quad \text{or} \quad \det\left[\overline{\mathbf{G}}_0 - (\eta + \mathrm{i}T_1)\mathbf{I}\right] = 0. \tag{71}$$

Using Eqs. (69) and (70), we find the four roots of Eq. (71) in the form

$$\eta = \frac{n\pi}{2}\left[-(\delta_i + \delta_e) \pm \mathrm{i}2n\Delta_\Omega \pm \sqrt{\left(\xi - \frac{\delta_i}{n}\right)^2 - \Delta_k^2}\right]. \tag{72}$$

It is easy to see that the asymptotic stability condition (36) for the roots in Eq. (72) reduces to the stability conditions of the non-resonant case (47) with $\Omega = 1/n$. Therefore, the first-order approximation does not predict extra instability regions near the resonant rotation speeds for $n \geq 2$.

One may notice some analogy of the resonances just described with the resonances of the Mathieu equation, where the secondary resonant zones are also degenerate in the first approximation, e.g., [18, Section 9.4]. However, all secondary instability zones exist for the Mathieu equation. These zones are very thin near the resonance points and require higher-order approximations in the perturbation theory to be understood analytically [27]. We have the same situation in our problem, as indeed confirmed by numerical simulations. Fig. 6 shows very thin instability zones, which appear for $n=2$, and similar zones were observed for larger n.

The first-order approximation can be used to understand some properties of the resonant zone in Fig. 6. Writing Eq. (72) for the case of no damping and contact forces, $\delta_i = \delta_e = \xi = 0$, we have

$$\eta = \frac{\mathrm{i}n\pi}{2}\left[\pm 2n\Delta_\Omega \pm \Delta_k\right] \tag{73}$$

with four different choices of the signs. When $\Delta_k > 0$, there are three conditions

$$\Delta_k = \pm 2n\Delta_\Omega \quad \text{or} \quad \Delta_\Omega = 0, \tag{74}$$

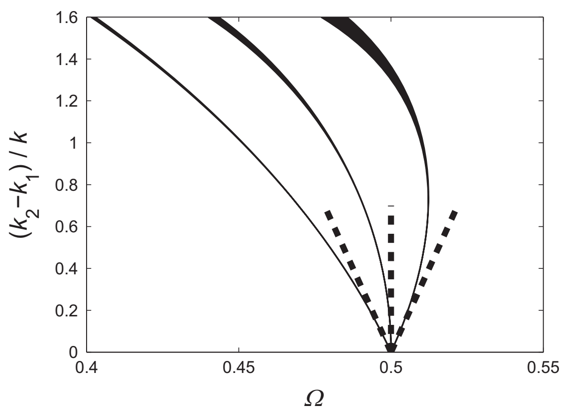

Fig. 6. Trefoil shape of the secondary resonant zone near $\Omega_0 = 1/2$ obtained using numerical simulations. Equal static and rotating asymmetries are considered, $c_1 = k_1$ and $c_2 = k_2$, with vanishing nonconservative forces, $\xi = \delta_i = \delta_e = 0$. Dashed lines show the directions of the resonant trefoil obtained analytically.

when two roots in Eq. (73) coincide leading to a double eigenvalue ρ in Eq. (34). In the presence of multiple eigenvalues, the system may become unstable due to higher order terms, see [18], as it occurs for our rotor. The three straight lines (74) are shown in Fig. 6 and determine the initial directions of the trefoil resonance zone.

The secondary resonant zones are very thin and highly sensitive to damping forces and, thus, they will be difficult to observe in realistic systems. On the other hand, these regions may get considerably larger and, thus, become technically important for nonuniform (oscillating) rotation originating, for example, from a misalignment of the rotation shaft. In this case the angular velocity Ω is not a constant, but a periodic function of time. Then, the system matrix (16) contains a full Fourier series and leads to nonzero matrices $\mathbf{G}_1$ similar to Eq. (53) for all n. As a result, the trefoil instability zones appear already in the first-order approximation for all secondary resonances.

We conclude that secondary resonant zones exist for the rotor in a way, which resembles the classical parametric resonance theory [18, Section 11.2]. However, there are fundamental differences related to much higher degeneracy (quadruple instead of a double eigenvalue) and, as a result, a new (trefoil instead of a tongue) shape of the resonant region. It is interesting that the secondary resonance regions appear in subcritical range and originate from the combined action of static and rotating asymmetries. Note that such regions cannot appear if equations of motion have constant coefficients in the rotating frame [10].

7. Conclusions

In this paper we studied a non-circular Laval rotor in asymmetric beddings under dissipative and follower force type of forcing. Deriving analytical approximations for the stability boundaries, an insight into the interactions of different effects could be obtained. Normally in technical systems asymmetry is small and the dissipative and follower forces are small compared to elastic restoring terms. This allowed us to treat asymmetries and nonconservative forces as perturbations of a perfectly symmetric rotor. Despite the asymmetry and nonconservative forcing being small we saw that they drastically influence the stability behavior. From a mathematical point of view, the problem is interesting because the unperturbed system has eigenvalues up to multiplicity four in the context of Floquet theory leading to complicated analysis and highly nonlinear form of stability conditions. From an engineering point of view, the problem is important because only by joint consideration of internal and external asymmetries a universal understanding of the major effects is possible.

The analysis reveals that the rotor equations with constant coefficients arising from formulations either in a rotating or in a stationary frame are degenerate since only one type of asymmetry can be studied at a time. From a design point of view, one should note that, for translatory degrees of freedom orthogonal to the axis of rotation, the increase of external asymmetry has a stabilizing effect in the sub- and supercritical ranges, which is not destroyed by internal asymmetry in the first approximation. In many engineering problems stiffness properties are much easier to design than for example damping, since the stiffness properties can normally be influenced by geometric designs. Especially around the critical speed, the stability behavior is extremely rich and even the increase of external asymmetry cannot completely remove instability zones. This once more reveals why the critical speed range in technical applications is passed by as quickly as possible when a rotor is accelerated or decelerated. Finally, the joint action of the rotor and bedding asymmetries induces secondary resonance zones in the subcritical region, which do not appear when the symmetries are considered independently.

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